

Identification and Estimation of Level- k Auctions

Ben Gillen*

California Institute of Technology

12 Aug 2010

Abstract

I develop a structural econometric framework for estimating first-price auctions with a level- k behavioral model, which nests the Bayesian Nash Equilibrium by allowing bidders to hold heterogeneous beliefs about opponents' bidding strategies. While behavioral heterogeneity causes identification of the joint distribution over valuations and bidder-types to fail under benchmark information sets, exclusion restrictions recover identification and allow testability of the behavioral model. I establish the nonparametric consistency of a semi-nonparametric maximum likelihood sieve estimator using Legendre polynomials by addressing irregularities in the population criterion function. An application characterizes the partial identification of expected revenues in auctions with a reserve price and of optimal reserve prices in incompletely identified models. An illustrative empirical analysis using data from USFS timber auctions finds a misspecified equilibrium optimal reserve price can substantially reduce expected revenues relative to an unbinding reserve price.

*Division of the Humanities and Social Sciences; MC 228-77; California Institute of Technology; Pasadena, CA 91125. This paper is taken from the first chapter of my doctoral thesis at the University of California, San Diego. I owe special thanks to Vincent Crawford and to my adviser Allan Timmermann for their guidance and detailed commentary on several drafts of the paper. I thank Philip Haile for providing data on USFS timber auctions. The paper has benefitted from significant comments and helpful feedback from Nageeb Ali, Yonghong An, Gray Calhoun, Graham Elliott, Michael Ewens, Hiroaki Kaido, Ivana Komunjer, Jacob LaRiviere, Xun Lu, David Miller, Andres Santos, Matt Shum, Joel Sobel, Suyong Song, Bryan Tomlin, Liang Choon Wang, Quang Vuong, and Hal White, along with seminar participants at UCSD, Stanford's SITE Conference, Caltech, Chicago's Booth School of Business, Maryland's Smith School of Business, the University of Miami, and the University of Wisconsin. Comments, feedback, and suggestions are welcome at bgillen@caltech.edu.

Contents

1	The Auction Model and Identification Objective	5
1.1	The Auction Model and Observational Equivalence	5
1.2	The Level- k Behavioral Model	7
1.3	Regularity Conditions	9
1.4	Example: Log-Normally Distributed Valuations	10
2	Nonparametric Identification in Homogeneous Populations	12
3	Nonparametric Identification in Heterogeneous Populations	15
3.1	Identification from Repeated Individual Observations	16
3.2	Identification under Pooled Bidding Behavior	17
4	Nonparametrically Consistent SNP Estimation	20
4.1	Monte Carlo Simulations	24
5	Expected Revenues & Optimal Reserve Pricing	26
6	Optimal Reserve Pricing in USFS Timber Auctions	27
7	Conclusion	32
	References	36
	Appendices	42
	A1 Proofs	42
	Not for Publication: Web Appendix	i
	W.1 Strategic Beliefs and Equilibrium Identification	i
	W.2 Identification Details for Level- k Behavioral Types	ii
	W.3 Semi-Nonparametric Maximum Likelihood Estimation Details	viii
	W.4 Level- k Bidding with Reserve Prices	xvi

Empirical studies of auctions focus on estimating the distribution over valuations held by a representative bidder for the item being sold. This distribution plays a key role in counterfactual analysis and in characterizing the effect of policy (such as the auction's reserve price) on the expected revenue from the sale. In the estimation problem, an econometrician interprets data on bidders' characteristics, the object for sale, and the bids themselves, using a structural econometric model to infer the population distribution over latent values. The model links an individual bidder's unobserved valuation to their observed bid by imposing structure on the dependence in valuations across bidders, individual risk preferences, and the strategic beliefs bidders hold when choosing their optimal bid. A substantial literature leverages the Bayesian Nash Equilibrium (BNE) model of behavior to simultaneously impose structure on behavior and beliefs that allows for the unique identification and optimal estimation of the distribution of latent bidder valuations from the observed distribution over bids. Seminal contributions in this line of research include Laffont and Vuong (1993), Donald and Paarsch (1996), Guerre, Perrigne, and Vuong (2000), and Athey and Haile (2002), with broad surveys in Athey and Haile (2005) and Paarsch and Hong (2006)'s textbook.

In this paper, I consider the empirical analysis of auctions when the Bayesian Nash Equilibrium behavioral model is nested within the level- k behavioral model proposed by Crawford and Iriberri (2007a) based on the theoretical developments in Costa-Gomes, Crawford, and Broseta (2001), Crawford (2003) and Camerer, Ho, and Chong (2004). The level- k model retains the equilibrium assumption that players best respond to beliefs about opponents' strategies but allows these beliefs to be heterogeneous and drawn from a structured hierarchy that gives rise to a mixture of behavioral types in the population. Beyond rationalizability, the level- k model restricts a player's bidding strategy to a set of decision rules, or types, defined through an iterated belief hierarchy anchored in an uninformative Level-0 (L0) model of opponents' behavior. The Level-1 (L1) player-type bids optimally based on the belief that all their opponents follow the L0 strategy, the Level-2 (L2) player best responds to the belief that all their opponents follow the L1 strategy, and so on. By nesting the BNE behavioral model within its hierarchy, the more general level- k framework provides a strategically robust foundation for inference that also provides information about the level of strategic sophistication in the population. As such, incorporating this additional dimension to the

model allows equilibrium to be tested against the directed alternative of a level- k behavioral model, providing external validation of the evidence accumulated in lab settings.

When Bayesian Nash Equilibrium is augmented with a level- k model of behavior, existing identification arguments no longer apply and identification generally fails in benchmark settings. In the BNE behavioral model, all individuals best respond to beliefs consistent with the observable distribution over bids, providing a key to the BNE identification argument that is not available in the level- k identification argument. The identification analysis begins by considering the setting in which the econometrician has sufficient information to identify the population distribution over bids in a homogeneous population of known bidder-types. This setting highlights the link between bids and valuations conditional on the behavioral type, while controlling for potential issues related to identifying the distribution over bidder-types. Having established identification in a homogeneous population of bidders, identification fails in heterogeneous populations due to the need to recover both the distribution over valuations and the distribution over behavioral types. This expanded model has a dimensionality that exceeds the dimensionality of the information set, resulting in an incompletely identified model under the benchmark specification where the econometrician observes only the distribution over bids. As a consequence, for any distribution over bidder-types, there exists a distribution over latent valuations consistent with the observed distribution over bids.

The problem I consider is closest to an application in Aradillas-Lopez and Tamer (2008), who present general results for identification in strategic models under level- k rationalizability, as defined by Bernheim (1984) and Pearce (1984). Aradillas-Lopez & Tamer's set identification result provides an upper bound on the cumulative distribution function for the distribution over valuations (since no one would rationally choose to bid more than they thought the item was worth), but they go on to show that the identified set also includes any sufficiently regular distribution that first order stochastically dominates this bound. The additional structure of Crawford and Iriberri (2007a)'s level- k auction model provides significant identification restrictions beyond those of rationalizability by placing an upper bound on an individual's bid shade and, consequently, a lower bound for the cumulative density function over valuations. More precisely, I show the identified set for the distribution over valuations in Crawford and Iriberri (2007a)'s model is isomorphic to the unit simplex repre-

senting the set of possible distributions over types and reduces to point identification in the presence of exclusion restrictions. By exploiting this additional structure, the level- k model provides a much tighter identified set than is available in Aradillas-Lopez & Tamer’s model, and admits point-identification for both the distribution over valuations and the distribution over bidder types under viable information specifications.

This paper contributes to the growing literature analyzing the econometrics of strategic models without imposing Bayesian Nash Equilibrium behavior. Among the first statistical models generalizing BNE, the Quantal Response Equilibrium (QRE) due to McKelvey and Palfrey (1995, 1998) provides a mechanism for incorporating statistical noise into individual behavior at games. Haile, Hortacsu, and Kosenok (2008) and Goeree, Holt, and Palfrey (2005) illustrate the need for structure on that noise to obtain empirical restrictions, but by exploiting the model’s generality, Rogers, Palfrey, and Camerer (2009) introduce a heterogeneous model of QRE with a structured error term that nests cognitive hierarchy behavior. Goeree, Holt, and Palfrey (2002) solve the QRE for auction models, illustrating QRE’s ability to generate overbidding in the presence of asymmetric distribution over valuations. Bajari and Hortacsu (2005) develop a structural econometric model for interpreting experimental auction data based on the QRE solution and also introduce an alternative non-equilibrium approach based on modeling behavior as an adaptive learning strategy. In analyzing behavior at English auctions with jump bidding, Haile and Tamer (2003) develop a model that only imposes individual behavior be rationalizable, which results in a partially identified distribution over valuations. Haile and Tamer (2003) derive tight bounds for this identified set and present empirical estimates for the set-identified optimal reserve price in US Forestry Service timber auctions.

While the seminal identification argument in Guerre, Perrigne, and Vuong (2000) leads to the natural derivation of an optimal indirect estimator, my identification results do not yield a clear estimation strategy for the level- k auction model outside of trivial settings. To address estimation of the model, I propose a consistent Semi-Nonparametric Maximum Likelihood (SNP-ML) estimator based on the Legendre polynomial sieve proposed by Bierens (2008) and applied to equilibrium auction models by Bierens and Song (2010). The mixture of bidder-types in level- k auctions gives rise to an upper semicontinuous likelihood criterion

function, requiring a more general uniform strong law of large numbers than applied by Bierens (2008) and by Bierens and Song (2010). I adopt a specialized version of the uniform strong law of large numbers developed in Artstein and Wets (1995), which is closely related to Hess (1996)'s results using proof techniques based on epiconvergence that have been particularly useful in the nascent study of set estimators for partially identified models.¹ This uniform strong law of large numbers provides the key to extending the parametric consistency results from Chernozhukov and Hong (2004) to a compact, infinite dimensional parameter space. Consistency of the SNP-ML estimator with upper semicontinuous likelihood functions also allows the econometrician to flexibly control for observed auction heterogeneity that may shift the support for the distribution over valuations, extending Donald and Paarsch (1996)'s maximum likelihood consistency result to an infinite dimensional semi-nonparametric distribution over valuations. As Donald and Paarsch (1996) show, this flexibility comes at the expense of non-standard asymptotic analysis that hinders characterizing the distribution of test statistics based on the SNP-ML estimator.

In an application of the econometric model, I consider the mechanism design problem of selecting the reserve price to maximize expected revenues in a level- k model. This analysis extends the level- k bidding model for auctions with a reserve price developed in Crawford, Kugler, Neeman, and Pauzner (2009) to a general distribution over valuations with more than two bidders participating. I also address the cases where the mechanism designer does not know the composition of the bidding population and, even worse, when the distribution over bidder-types is unidentified. Solving for the expected revenue to the seller at a fixed reserve price, I present first order conditions for the optimal reserve price similar to those developed by Myerson (1981). In the setting where the distribution over bidder-types is not identified, the seller's expected revenues are only partially identified, belonging to a closed, convex set. As such, identification of the optimal reserve price is similar to the result in Haile and Tamer (2003)'s study of jump bidding in English auctions, with the optimal reserve price in the unidentified model belonging to a compact set.

To illustrate estimation, I implement the model using data from USFS timber auctions

¹Kaido (2009) uses weak epiconvergence arguments in deriving distributional results that unify set estimation techniques based on vector support functions developed by Beresteanu and Molinari (2008) and approximate minimizers to criterion functions developed by Chernozhukov, Hong, and Tamer (2007).

that has been widely analyzed in empirical auction studies. Given simulation evidence that empirically separating bidder-types is practically quite difficult, the application focuses on the properties of inference and counterfactual analysis with an unidentified behavioral model model. The empirical results illustrate that, even though the distribution over valuations may be fairly tightly identified, the ambiguous distribution over bidder-types has a substantial implication on counterfactual analysis. Contrary to the equilibrium implication that introducing a binding reserve price increases expected revenues, given a sufficiently large population of unsophisticated bidders, expected revenues are maximized by setting the reserve price equal to the government’s appraised value for the tract. This result rationalizes the current USFS policy of implementing non-binding reserve prices without having to appeal to non-revenue motives.

The paper proceeds as follows. Section 1 introduces the auction model and identification problem, illustrating the effect of behavioral misspecification on inference in a simple parametric example. Section 2 addresses identification in homogeneous populations, presenting an omnibus identification theorem with conditions under which a general level- k model when all bidders follow the same behavioral rule. I extend these identification arguments to heterogeneous populations in section 3, beginning with the setting where each individual’s type is observed by the econometrician and exploring exclusion restrictions that identify the distribution over types. Section 4 discusses estimation of the model in a semi-nonparametric framework, providing an overview of the argument establishing nonparametric consistency for the estimator. Section 5 addresses the mechanism design problem, with section 6 presenting estimates for optimal reserve pricing from USFS timber auctions before concluding.

1 The Auction Model and Identification Objective

1.1 The Auction Model and Observational Equivalence

I consider identification in the risk neutral symmetric Independent Private Values (IPV) specification of the general Milgrom and Weber (1982) first price auction model.² In the

²In the second-price auction format with Independent Private Values, equilibrium is in weakly dominant strategies so that, for any belief of opponent’s behavior, selecting a bid equal to the agent’s valuation is a

period t IPV auction, each player $i \in N_t \in \mathbb{Z}$ observes N_t and the realization of a random variable, X_{it} , which has a commonly known distribution, $F_X(x)$, that is absolutely continuous over $[0, \bar{x}] \subset \mathbb{R}^+$ with a strictly positive pdf $f_X(x)$. The variable $X_{i,t}$ specifies bidder i 's latent valuation for the good being sold at auction t . Bidder i then chooses his bid, s_{it} , conditional on this valuation and pays the value of his bid in exchange for the good if he submits the highest bid in the auction. Given that f is continuously differentiable and bounded away from zero, Maskin and Riley (1984) show that there exists an equilibrium in strictly monotonic and continuously differentiable bidding strategies.

As is standard in auction identification problems, the econometrician's benchmark information set consists of the empirical distribution over bids, denoted $F_S(s)$.³ When the inverse bidding function exists, this distribution over bids is generated by the true distribution over valuations and the equilibrium bidding function, $\sigma_{Eqm,X}(x)$, when $F_S(s) = F_X(\sigma_{Eqm,X}^{-1}(s))$. The econometric model under equilibrium is defined entirely by the distribution over valuations, which is identified if it is the unique distribution that generates the observed distribution over bids. Given this definition, identification fails if there exists an alternative distribution over valuations, $F^* \neq F_X$, that is observationally equivalent to F_X , defined as:

Definition 1 (Equilibrium Observational Equivalence) *A structure (F_X) coupled with the equilibrium bidding rule $\sigma_{Eqm,X}(x)$ is observationally equivalent to the structure (F_*) coupled with the corresponding equilibrium bidding rule $\sigma_{Eqm,*}(x)$ if:*

$$F_X(\sigma_{Eqm,X}^{-1}(s)) = F_S(s) = F_*(\sigma_{Eqm,*}^{-1}(s))$$

The seminal identification argument in Guerre, Perrigne, and Vuong (2000) analyzes the best response. Since the level- k bidding behavior is identical to the equilibrium weakly-dominant strategy in this setting, identification is inherited from existing results and the distribution over bidder-types is trivially non-identified. The independence and private valuation assumption avoids known challenges to identification from Athey and Haile (2002) in affiliated values models, leaving these problems for future work. Addressing the pure common values model that Fevrier (2008) studies under equilibrium would highlight the empirical relationship between the level- k behavioral model and the notion of "cursedness" in Eyster and Rabin (2005). Symmetry is assumed largely for tractability and can be relaxed by extending Brendstrup and Paarsch (2006). Common knowledge of the number of bidders in each auction provides a key to identification that focuses on the strategic trade-offs an individual makes when confronted with different levels of competition.

³Note that, as a consequence of independent bidding, the distribution $F_S(s)$ is identified even if the econometrician only observes the winning bid in the auction.

bid function transforming valuations into bids to establish that all observationally equivalent distributions over valuations are identical. The technical web appendix includes a slightly more direct proof of this result that focuses on strategic beliefs to highlight the equilibrium property central to identification: player i 's expected utility depends only on their own independent private valuation and the distribution of other player's actions, which is contained in the econometrician's information set. To summarize the auction model:

Assumption 1 (IPV Auction Model) *Unless explicitly stated otherwise,*

- a. $U_i(X_{it}, s_{1t}, \dots, s_{Nt}) = (X_{it} - s_{it}) 1_{\{s_{it} > \max_{j \neq i} s_{jt}\}}$
- b. $X_{it} \sim_{iid} F_X$ which is absolutely continuous over $\mathcal{X} = [0, \bar{x}] \subset \mathbb{R}^+$ with strictly positive pdf $f_X(x)$
- c. N_t and F_X are common knowledge.

1.2 The Level- k Behavioral Model

In Crawford and Iriberri (2007a)'s level- k auction model, a player draws their beliefs about other players' actions from a cognitive hierarchy and best responds to those beliefs, giving rise to a mixture of heterogeneous behavioral types in the population.⁴ The level- k behavioral model provides a parsimonious framework of how people make decisions in novel strategic settings by imposing structure on the possible strategic beliefs they hold when choosing their optimal behavior. Crawford & Iriberri consider two specifications for anchoring level zero (L0) beliefs and characterize the strategies for L1 and L2 players based on these anchoring beliefs. Players with higher levels of sophistication are not observed in the lab and, as such, are treated as effectively non-existent. A Random L0 player ($L0_R$) bids uniformly over the

⁴The level- k model's mixture-of-types framework is rooted in earlier work by a number of researchers, including Harless and Camerer (1994), Stahl and Wilson (1995), Nagel (1995), and El Gamal and Grether (1995) with additional theoretical development by Costa-Gomes, Crawford, and Broseta (2001) and Camerer, Ho, and Chong (2004). Ho, Camerer, and Weigelt (1998) and Bosch-Domenech, Montalvo, Nagel, and Satorra (2002) apply the model to analyzing behavior in beauty contest games, presenting some of the first evidence that players rarely reach beyond the second level of the strategic hierarchy. Crawford (2003), Costa-Gomes and Crawford (2006), and Crawford and Iriberri (2007b) explore a number of applications for the model relating to strategic search and information transmission. Ivanov, Levin, and Niederle (2010) present experimental evidence that presents some skepticism regarding the extent to which individuals follow rationalizable rules in choosing their behavior, illustrating the difficulty in controlling individual beliefs.

set of possible valuations and a Truthful L0 player ($L0_T$) truthfully reveals their expected valuation for the object on auction. These two anchoring beliefs each give rise to two belief hierarchies and behavioral types, with level-type Lk_τ best responding to the belief that everyone plays the $L(k-1)_\tau$ strategy. When coupled with the equilibrium strategy behavior (corresponding to a level- ∞ bidder type), the hierarchy allows for seven potentially different behavioral types with two of those types assumed to exist only in the imagination of other players. To summarize the behavioral assumptions:

Assumption 2 (Level- k Behavioral Model) *In the level- k auction model,*

- a. *a player observes their valuation, the number of bidders participating in the auction, and is assigned to a bidder-type k ,*
- b. *the player of type k 's strategy best responds to the belief that every other player plays according to the type $k-1$ strategy*
- c. *the level-0 bidder-types follow an uninformative strategy: the $L0_R$ bidder-type bids uniformly over the set of valuations and the $L0_T$ bidder-type bids their valuation, and,*
- d. *the level- ∞ bidder-type bids in accordance with the BNE strategy.*

A similar approach to modeling strategic behavior is proposed in Camerer, Ho, and Chong (2004), who develop a cognitive hierarchy model for one-shot games rooted on an uninformative level-0 behavioral model. In contrast to Costa-Gomes, Crawford, and Broseta (2001)'s level- k behavioral model, players at the k -th level in the cognitive hierarchy do not believe that every other player follows the level $k-1$ strategy, but rather believes there is a mixture of players following the level-0 through level $k-1$ strategies. As such, while the k -th level behavioral type in the cognitive hierarchy is oblivious to people playing at their own or higher levels of sophistication, they recognize heterogeneity among lower bidder-types and know the relative proportion of the lower-level bidding types. In a model that assumes no level-0 types exist in the population, the level-1 and level-2 cognitive hierarchy behavioral types are identical to the $L1_R$ and $L2_R$ bidder types defined above, with the only difference between the models realized at higher levels of sophistication. Experimental evidence for

the IPV setting presented in Crawford and Iriberri (2007a)’s online appendix suggests the truthful hierarchy of types are not as prominent under the IPV model as in common values settings. For this reason, the application only considers estimating the hierarchy based on the Random Level-0 bidder-type. As such, the only applied difference in the two modeling approaches arises for the level- ∞ bidder-type, who follows the BNE strategy in the level- k model but best responds to the empirical distribution over bids in the cognitive hierarchy model.⁵

1.3 Regularity Conditions

Regularity conditions on the distribution over valuations ensure the agents’ bidding functions are strictly monotonic so as to preclude pooling behavior that would stymie identification and introduce atoms to the estimation problem. While Maskin and Riley (1984) show these conditions can be quite weak under equilibrium, the level- k model requires some additional restrictions to ensure the bidding functions are well-behaved. These conditions are similar to Myerson (1981)’s regularity condition that ensures the mechanism designer’s revenue optimizing problem is well defined:

$$1 - \frac{d}{dx} \frac{1 - F_X(x)}{f_X(x)} > 0 \tag{1}$$

Myerson’s regularity condition presents a restriction on the inverse hazard rate for the event that one bidder will have a valuation exceeding a reservation price. Ensuring the level- k bidding strategy is continuously differentiable and strictly monotonic requires a restriction on the inverse hazard rate for the event that the player would win the auction. These conditions strengthen regularity conditions in standard auction theory, with assumption L1.3 representing the level- k analog to the Myerson’s regularity condition. For purposes of generality, the lemma places conditions on the level $(k - 1)$ bidding strategy, though it is

⁵Camerer, Ho, & Chong also introduce additional structure by assuming the sophistication of a bidder is drawn from a Poisson distribution, which captures the intuition of the cognitive hierarchy as arising from an iterative reasoning procedure. Given the identification results, this Poisson Cognitive Hierarchy model is a restricted version of an unrestricted distribution over bidder-types. Computationally, the level- ∞ bidder-type under the cognitive hierarchy is easier to accommodate as their beliefs can be calculated directly from the empirical distribution over bids using a strategy similar to Guerre, Perrigne, and Vuong (2000).

straight-forward to verify these conditions hold inductively as a condition on the primitive distribution over valuations.

Lemma 1 (Level- k Regularity Conditions) *Suppose:*

L1.1 *The level- $(k - 1)$ bidding strategy is a twice continuously differentiable, strictly monotonic function of their valuations.*

L1.2 *F_X has k continuous, bounded derivatives over support $[0, \bar{x}]$, and*

L1.3 *$\exists \xi > 0$ such that, for all possible valuations $x \in \mathcal{X}$,*

$$1 - \frac{d}{dx} \frac{F_X(\sigma_{k-1}^{-1}(x))}{f_X(\sigma_{k-1}^{-1}(x)) \frac{d}{ds} \sigma_{k-1}^{-1}(s) |_{s=x}} > \xi \quad (2)$$

Then the level- k bidding strategy, $\sigma_k(x)$ is a bounded, strictly monotonic, and continuously differentiable function of the bidder's valuation.

The proof of Lemma 1 consists of implicitly differentiating the first order conditions of the level- k bidder-type's optimization problem. The regularity condition in equation 2 ensures the derivative of the bidding function is continuous and bounded away from zero by ξ , ensuring its invertability.

1.4 Example: Log-Normally Distributed Valuations

To illustrate the differences between the level- k and equilibrium auction models, consider an auction where $N = 4$ and valuations are log-normally distributed with zero mean and unit variance. Each player observes their valuations and is independently assigned to one of five behavioral types: Random Level-1 ($L1_R$), Random Level-2 ($L2_R$), Truthful Level-1 ($L1_T$), Truthful Level-2 ($L2_T$), or Equilibrium (Eqm). Figure 1 plots the behavioral strategies for each of these types, all of which are strictly monotonic and continuous over their support.

Figure 2 plots the distribution of bids for the different bidder types. Note that each of the types can be clearly differentiated from one another via a single-crossing property in the tail so that, given the bidding distributions for every agent, the econometrician would

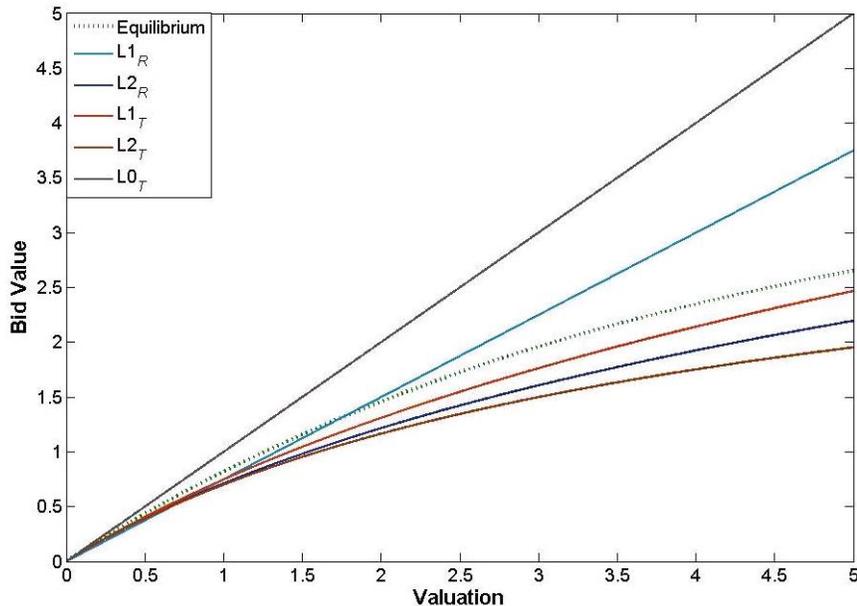


Figure 1: Monotonic Bidding Functions for Level-k Behavioral Types

Each bidder-type corresponds to a unique, monotonic, continuous bidding strategy. Further, each pair of bidding strategies satisfies a single-crossing property that allows the econometrician to separate them.

have sufficient information to sort each agent into their respective bidder-type. In addition, figure 2 includes the unconditional bid distribution for a sample population where a bidder is assigned to bidder-types $L1_T$, $L2_T$, $L1_R$, $L2_R$, and Eqm with probabilities 60%, 5%, 15%, 5%, and 15%, respectively. The distribution over bids from this sample population represents the econometrician's benchmark information set from the auction.

What if the econometrician ignored the heterogeneous behavior, and instead estimated the distribution over valuations using a mis-specified equilibrium model of behavior? In Figure 2, the equilibrium bidder-type's distribution over bids has significantly thinner tails than the sample distribution, so the mis-specified distribution over valuations would imply much fatter tails than the truth. Figure 3 illustrates that the magnitude of this misspecification effect can be severe. The dashed line presents the true distribution of valuations, the solid green line presents the distribution of the winning bid, and the solid red line presents the estimated distribution of valuations with mis-specification. While the true 95th quantile valuation is only 5, the estimated 95th quantile is an order of magnitude larger, extending

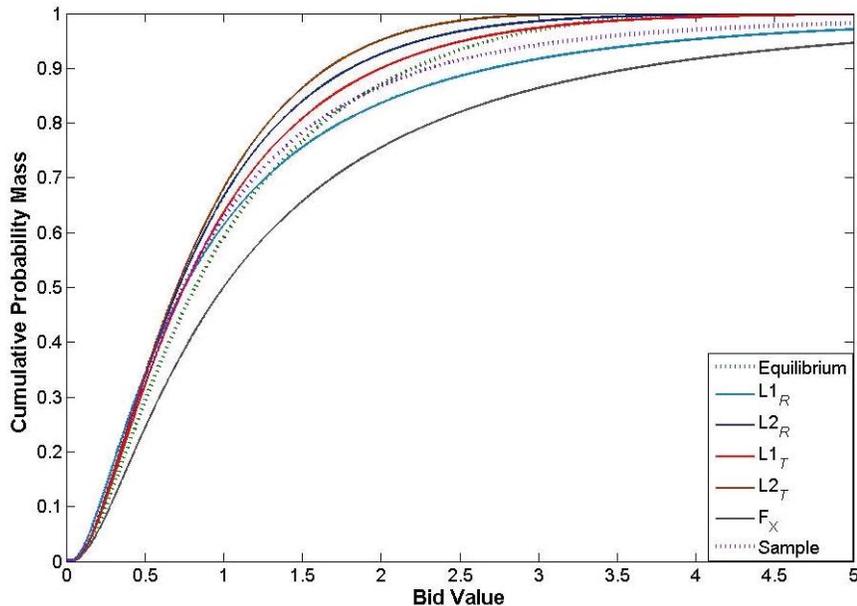


Figure 2: Distribution over Bids for Level-k Behavioral Types

Each bidder-type corresponds to a unique distribution over bids. The sample distribution over bids is the mixture of these distributions that is observed by the econometrician. Note that the tail of the Sample distribution is substantially fatter than the tail for the distribution associated with the equilibrium bidder-type, so we'd expect estimation based on the equilibrium model to recover a distribution over valuations with substantially fatter tails.

beyond the boundary of the graph to nearly 50. This bias is driven by overbidding from lower-level types that, under equilibrium, can only be justified by large valuations.

2 Nonparametric Identification in Homogeneous Populations

Characterizing non-parametric identification in the level- k auction model requires first establishing identification when all players behave homogeneously, following the strategy of a single bidder-type. The following theorem, proved in Appendix 1, generally characterizes identification of a level- k bidder-type in homogeneous populations. A technical web appendix walks through the detailed identification arguments for each of the behavioral types in the Crawford and Iriberri (2007a) model, developing the intuition behind the proof in

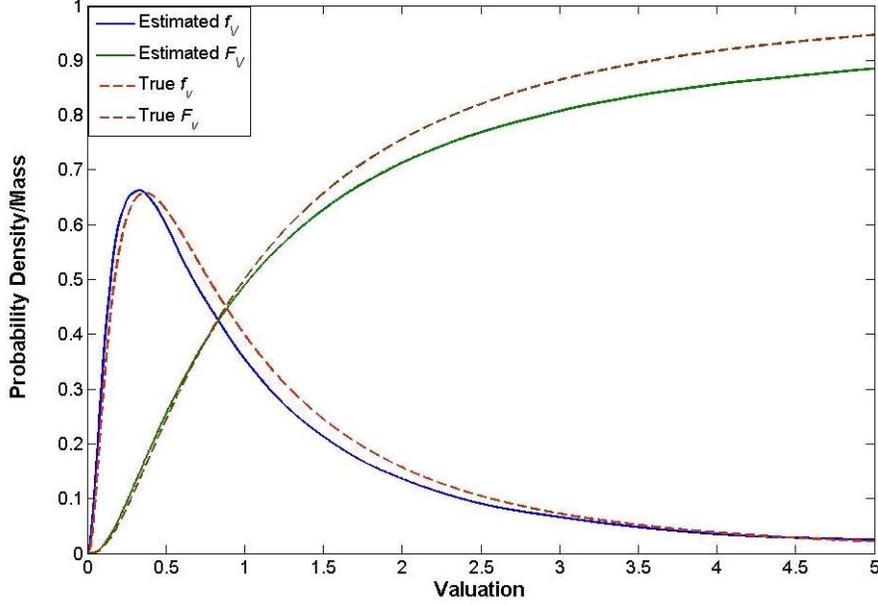


Figure 3: Incorrect Inference Caused by Behavioral Misspecification

Assuming an equilibrium behavioral model in the presence of behavioral bidders can lead to substantial errors in inference. The estimated distribution represented by solid lines has a substantially fatter tail than the true distribution represented by the dashed line.

progressively more complicated settings.

Theorem 1 (Identification of Level- k Auctions with Homogeneous Populations)

Assume the conditions of Assumptions 1 and 2 and Lemma 1. Suppose the econometrician observes the distribution over bids, $F_{S_{N,k}}$, for a homogeneous population of level- k bidders with N bidders participating in each auction, then the distribution F_X is uniquely identified.

Suppose further that the econometrician observes the distribution over bids, $F_{S_{N^,k}}$, for a homogeneous population of level- k bidders with $N^* \neq N$ bidders participating in each auction, then the level- k behavioral model is testable through overidentifying restrictions.*

This result presents a refinement of Aradillas-Lopez and Tamer (2008)’s Proposition 2 by establishing conditions for a pointwise mapping between strategic beliefs coupled with the distribution over bids and the implied distribution over valuations. Whereas Aradillas-Lopez & Tamer show that a large set of rationalizable beliefs yields a large set of possible distributions over valuations, theorem 1 shows that, under suitable conditions, any single

rationalizable belief yields a single implied distribution over valuations.

Before discussing identification in heterogeneous populations, I first consider overidentifying restrictions that could reject the level- k behavioral model as mis-specified. As in the equilibrium model, if the econometrician always observes auctions with a fixed number of participating bidders (for example, if every auction has exactly 5 bidders), the level- k behavioral model imposes no testable restrictions on the data beyond those implied by independence of individual bidding decisions and those required for identification. That is, for any distribution over bids, \hat{F}_S , there exists a corresponding distribution over valuations, \hat{F}_{X,Lk_τ} , admitting a strictly monotonic inverse bidding function consistent with the hypothesis that the entire population of bidders is of type Lk_τ . This result follows from theorem 1, which holds for any sufficiently regular bidder-type and distribution over bids.

In another parallel to equilibrium results, though, the level- k model is testable if the number of participating bidders in the auction varies exogenously (for example, if half of the auctions in the sample have 5 competing bidders and half have 20 bidders). The level- k behavioral model defines precisely how a bidder's strategy reacts to changes in the number of bidders participating in an auction, imposing a continuum of over-identifying restrictions on each quantile of the distribution over valuations. Suppose the econometrician observes just two distributions over bids corresponding to two different levels of competition in the auction, $\hat{F}_{S_{N_1}}$ and $\hat{F}_{S_{N_2}}$, and wishes to test the hypothesis that the entire population of bidders is of type Lk_τ . The econometrician can use the two distributions over bids to recover two distribution over valuations, \hat{F}_{X,Lk_τ,N_1} and \hat{F}_{X,Lk_τ,N_2} . If the hypothesized behavioral model is true, these two recovered distributions must be equal, that is, $\hat{F}_{X,Lk_\tau,N_1}(x) = \hat{F}_{X,Lk_\tau,N_2}(x)$, for all x . If any of the quantiles from the two distributions disagree, then the hypothesized behavioral model can be rejected. The model's testability applies not only to hypotheses that the population is homogeneous, but can also be used to test hypotheses about mixtures of bidder types, providing the basis for identification in heterogeneous populations.⁶

⁶In another result from Athey and Haile (2002) and Haile, Hong, and Shum (2003), given observation of more than one bid from each auction, different order statistics from the distribution over bids can test implications of the IPV model. For example, given the winning bid and the second highest bid, the econometrician could estimate the distribution over valuations from both samples of data individually and test the hypothesis that the estimated distributions are equivalent up to sampling error. As an implication of the independence of information and beliefs, the restriction would be violated in the affiliated values problem or in the presence of unobserved heterogeneity. However, since this independence also obtains under the level- k

3 Nonparametric Identification in Heterogeneous Populations

In heterogeneous populations, the need to identify the distribution over bidder-types in addition to the distribution over valuations introduces a free dimension to the model that requires additional information for identification. The expanded model with heterogeneous bidder types requires some notation regarding the population distribution over bidder-types and an independence assumption for bidder-type assignment. Define the set of K possible bidder-types by \mathcal{K} and denote the distribution over bidder-types, $p = [p_1, \dots, p_K]$, so that the probability that a bidder is of type k is p_k . Further, assume the assignment of bidder-types is independent of that individual's valuation.

Assumption 3 (Independent Assignment to Bidder-Types)

Each player $i \in N$ is randomly assigned a unique bidder-type $\tau(i) \in \mathcal{K}$ according to the distribution $p = [p_1, \dots, p_K]$ independently of the number of bidders in the auction and the player's latent valuation.

The econometric structure now consists of the true distribution over valuations, F_X , the set of behavioral types, \mathcal{K} , and the distribution over behavioral types, p . Since the set of behavioral types is defined by the economic theory, treat \mathcal{K} as known. The generalized definition of observational equivalence in heterogeneous populations becomes:

Definition 2 (Level- k Observational Equivalence)

Given the set of bidder types, a structure (F_X, \mathcal{K}, p) is observationally equivalent to the structure (F_, \mathcal{K}, p_*) if:*

$$\sum_{k=1}^K p_{X,k}(s) F_X(\sigma_{k,X}^{-1}(s)) = F_S(s) = \sum_{k=1}^K p_{*,k}(s) F_*(\sigma_{k,*}^{-1}(s))$$

where, $p_{*,k}(s) = \frac{p_{*,k} F_*(\sigma_{k,*}^{-1}(s))}{\sum_{\kappa=1}^K p_{*,\kappa} F_*(\sigma_{\kappa,*}^{-1}(s))}$

model, this test would have no power against behavioral mis-specification.

3.1 Identification from Repeated Individual Observations

Suppose the population of bidder-types is constant in each auction and the econometrician observes each individual's bidding behavior across a large number of independent auctions. This information is sufficient to characterize each individual's distribution over bids and, consequently, separate all bidders into K groups of bidder-types. All that remains to establish identification in this setting is to uniquely sort each of the observed bidder-types to a position in the behavioral hierarchy.

In Crawford and Iriberri (2007a)'s model, $\mathcal{K} = \{L0_T, L1_T, L2_T, L0_R, L1_R, L2_R, Eqm\}$, and the $L0_T$ bidder-type is identified as the bidder with the distribution over bids that first-order stochastically dominates all other bidder types' distributions. This information is sufficient to identify the distribution over valuations, which in turn identifies the bidder-types corresponding to each of the other distributions over bids by computing the implied distribution over bids for each bidder-type and matching it to the population distribution over bids. If there are no truthful bidders, the $L1_R$ bidder type's linear strategy identifies this bidder-type's distribution as the bidder with the thinnest right tail. This result is due to Battigalli and Siniscalchi (2003), who show that best responding to iteratively rationalizable bidding functions yields bidding functions that are generally concave and weakly decreasing in the iterations. In models with a unique Level-0 bidder-type, the set of recovered distributions over bids can be sorted into bidder-types based on their upper support. In particular, if the level $k - 1$ bidder-type never bids above \bar{s}_{k-1} , then the level k bidder's support will be weakly smaller than \bar{s}_{k-1} . Once a single distribution over bids is assigned to the bidder-type generating that distribution, the distribution over valuations is identified based on this association, an algorithm that applies due to the finite number of bidder-types that is not available to Aradillas-Lopez and Tamer (2008). In summary:

Theorem 2 (Identification with Repeated Individual Observations) *Suppose:*

1. *The econometrician observes distribution over bids for each individual in a heterogeneous population of level- k bidders responding to a total population of N possible bidders, and*
2. *The distribution over valuations is identified for each bidder-type $k \in \mathcal{K}$ from the distribution over bids, $F_{S,k}$, observed in homogeneous populations,*

then the distribution F_X is uniquely identified. Suppose further that:

3. *The econometrician observes bidding behavior for a homogeneous population of level- k bidders responding to a total population of $N^* \neq N$ possible bidders in each auction*

then the level- k behavioral model and the distribution over bidder-types are jointly testable.

Proof. The information set for the econometrician consists of the K distributions over bids, $F_{S,\tau_1}, \dots, F_{S,\tau_K}$. The identification task is then to assign these types (τ_1, \dots, τ_K) to one of the $K!$ possible permutations of true types. The restrictions that accomplish this task are generated by the fact that the distribution over valuations is constant for each of the behavioral types.

Suppose τ_1 is known, then the distribution over valuations $F_{X_{\tau_1}}$ is identified from F_{S,τ_1} . For the true value of τ_2 and any bid value s :

$$F_{S,\tau_2}(s) = F_{X_{\tau_1}}(\sigma_{\tau_2}^{-1}(s)) = F_{S,\tau_1}(\sigma_{\tau_1}(\sigma_{\tau_2}^{-1}(s))) \quad (3)$$

Identification fails if there are two values of τ_2 that satisfy equation 3. In this case, the distribution over bids must be the same for both types and, as such, their bidding functions must be identical so that one of the two types is redundant in the behavioral model. If there does not exist a compatible sort, then the data rejects the level- k model as mis-specified.

The testable implications for the cognitive hierarchy model with variation in the number of bidders are analogous to those established in Athey and Haile (2002) and discussed in section 2. As the number of bidders changes, the bidder-type's strategies change deterministically so that, having estimated F_X in a setting with N_1 bidders, the distribution over bids for N_2 bidders, F_{S,N_2} , is completely determined. As such, every quantile of F_{S,N_2} provides a testable restriction of the level- k model. ■

3.2 Identification under Pooled Bidding Behavior

The econometrician rarely observes the information set studied in the previous subsection. In the rare cases that individual bidding data is obtainable (for example, in sealed-bid auctions), anonymity concerns typically prevent tracking an individual bidder across auctions

and repeated interactions among bidders demands additional strategic analysis. More commonly, the econometrician is capable of observing the bids of all individuals in the population without being able to follow them from one auction to the other. In this benchmark informational setting, where the econometrician only observes sufficient information to identify the population distribution over bids, the model is incompletely identified.⁷ For example, the econometrician generally does not have sufficient information to differentiate whether the distribution over bids was generated by a population consisting entirely of truthful bidder-types or entirely of $L1_R$ bidder-types. The next theorem establishes the incomplete identification result as even more severe in that, for any given distribution over bids and any distribution over types, there exists a distribution over valuations that generates the observed distribution over bids.

Theorem 3 (Partial Identification in Heterogeneous Populations)

Suppose the econometrician observes $F_{S_N}(s)$, the distribution over bids in a heterogeneous population of N bidders, and that the distribution over valuations is identified for each bidder-type $k \in \mathcal{K}$ from F_{S_N} in homogeneous populations. Then, for any distribution over behavioral types, $p = [p_1, \dots, p_K]$, there exists a distribution $\tilde{F}_p(x)$ generating $F_{S_N}(s)$.

The proof in the appendix exploits the structure imposed by equation 3 to separate the mixture distribution $F_{S_N}(s)$ into its component distributions over bids for homogeneous populations. Having recovered the implied distributions of bids from homogeneous bidder-types, Theorem 2 proves that there exists a unique distribution over valuations that generates the recovered distributions over bids. The requirement that the distribution over valuations is constant across bidder-types is enforced through the decomposition of the mixture distribution over bids in the first step. The unique distribution over latent valuations for any and

⁷The identification results here are analogous to the equilibrium results when behavior is subject to unknown risk aversion. Campo, Guerre, Perrigne, and Vuong (Forthcoming) present the non-identification result for the benchmark equilibrium model under parametric HARA utility specification with an unknown risk aversion coefficient. In this case, it is possible to identify a distribution over valuations from the observed distribution over bids for any value of the risk aversion coefficient, resulting in a partially identified model. This partial identification result requires additional information to identify risk aversion and is closely related to the need for additional information to identify the distribution over types in a level- k auction model. The approach here follows Bajari and Hortacsu (2005), Perrigne and Vuong (2007), and Guerre, Perrigne, and Vuong (2009), who recover identification of the bidder’s utility function and the distribution over valuations by exploiting testable restrictions originally proposed by Athey and Haile (2002).

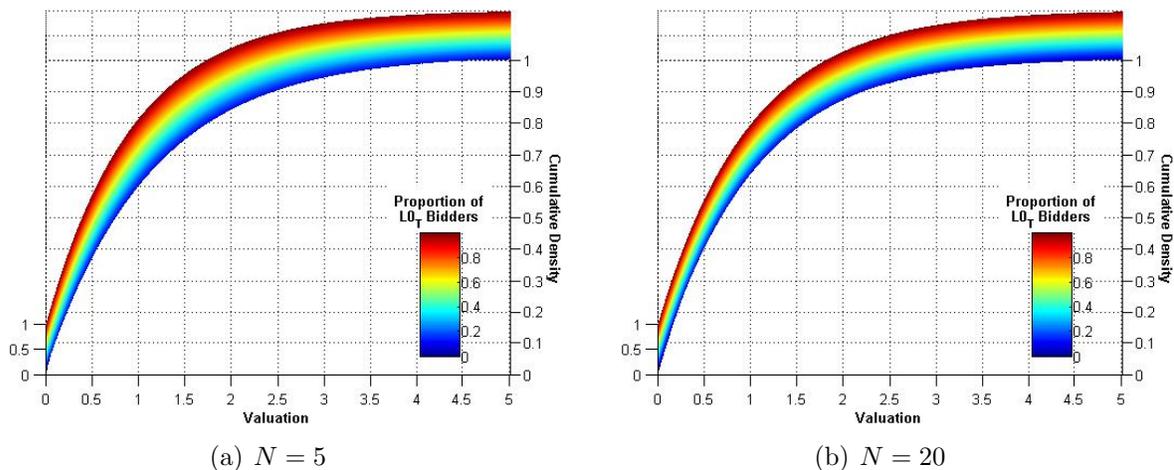


Figure 4: Identified Set when $\mathcal{K} = \{L0_T, L1_R\}$

The identified set when $N = 5$ and $N = 20$ bidders with an unknown distribution over Level-0 Truthful and Level-1 Random bidder-types includes a unique distribution over valuations for any mixture of the bidder-types. Note that the identified set shifts with the number of bidders for every estimated distribution except the true distribution over valuations, corresponding to the true distribution over bidder-types.

every hypothesized mixture of bidder-types establishes the partial identification result.

Figure 4 illustrates this result in a very simple setting with $\mathcal{K} = \{L0_T, L1_R\}$, $p_{L0_T} = 0.7$, and $p_{L1_R} = 0.3$ when the true valuations are exponentially distributed with either $N = 5$ or $N = 20$ bidders participating in the auction. Without knowing p , the true distribution of valuations could correspond to any one of the cdf's in the figure, with the z -axis providing depth to indicate the mixture of bidder-types that generates the distribution over bids from the hypothesized distribution over valuations. While the correct distribution of bidder-types recovers the true distribution of valuations, other distributions of bidder-types are incorrectly estimated under mis-specified behavioral models. Without additional information, the behavioral model only characterizes this identified set of observationally equivalent distributions over valuations.

Since strategic uncertainty changes with the level of competition, testable restrictions generated by exogenous variation in the number of bidders participating in each auction can provide that identifying information. Note the differences in the identified set for distributions over valuations when either 5 or 20 bidders participate in the auction in Figure 4. Since the underlying model is constant across the two settings, the true distribution over bidder-types and valuations must belong to both identified sets and any distribution that

does not belong to both sets can be rejected. Further, if the intersection of the identified sets for the distribution over valuations in the two populations is empty, then the level- k behavioral model is rejected. With a continuum of testible restrictions, only two different levels of competition are necessary to point identify the finite-dimensional distribution over types and the distribution over valuations.

The identification argument is illustrated in Figure 5, which intersects the identified set for $N = 5$ with the identified set for $N = 20$. The intersection selects the unique estimated distribution over valuations consistent across changes in the number of bidders corresponding to the true distribution over bidder types. If, however, variation in the number of bidders also results in variation in the distribution over bidder-types or shifts in the distribution over valuations, then the partial identification result from Theorem 3 applies. The following theorem summarizes this result:

Theorem 4 (Identification with Variation in Number of Bidders) *Suppose:*

1. *The econometrician observes the distributions over bids, $F_{S_{N_1}}$ and $F_{S_{N_2}}$, for a heterogeneous population of level- k bidders responding to N_1 and N_2 bidders with $N_1 \neq N_2$,*
2. *The distribution over valuations is identified for each bidder-type $k \in \mathcal{K}$ from the distribution over bids in homogeneous populations, and*
3. *For any $k_i, k_j \in \mathcal{K}$ with $k_i \neq k_j$, the set $\{x \in [0, \bar{x}] \mid \sigma_{k_i, N_1}(x) - \sigma_{k_i, N_2}(x) \neq \sigma_{k_j, N_1}(x) - \sigma_{k_j, N_2}(x)\}$ has nonzero Lebesgue measure.*

then both the distributions over valuations, F_X , and over behavioral types, p , are identified. Further, the level- k behavioral model imposes testable overidentifying restrictions.

4 Nonparametrically Consistent SNP Estimation

Assume the distribution over valuations is drawn from a compact family of distributions indexed by the infinite-dimensional parameter vector θ so that the true model is characterized by the true distributions both over valuations and bidder-types, (θ_0, p_0) . A web-based technical appendix derives the population log-likelihood criterion function and presents a detailed

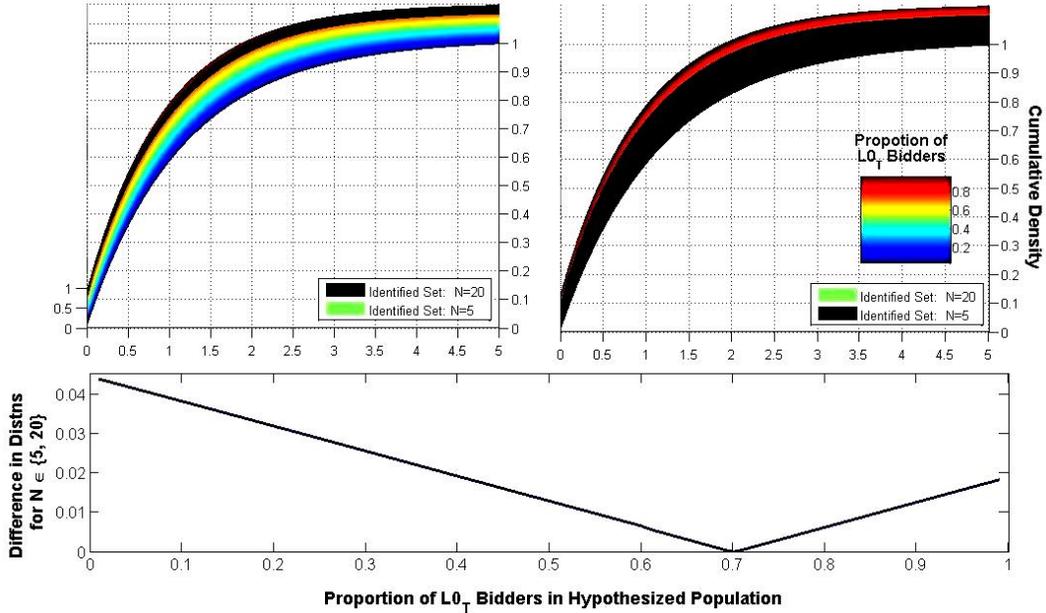


Figure 5: Identification from Intersecting Identified Sets for $N \in \{5, 20\}$

The top two panels show that when we plot the identified sets corresponding $N = 5$ bidders and $N = 20$ bidders, the intersection is unique. The bottom panel shows that this intersection (where the sup-norm of the difference between the distributions is zero) identifies the true distribution over bidder-types.

semi-nonparametric consistency argument for maximum likelihood estimation following from techniques pioneered by Gallant and Nychka (1987) and surveyed in Chen (2007). Beyond identification, consistent SNP-ML estimation relies on three conditions: a compact parameter space, a Uniform Strong Law of Large Numbers (USLLN) for the criterion function, and a truncation algorithm. This section provides a brief overview of these conditions, introducing some additional assumptions to ensure they are satisfied by the estimator.

Estimation assumes the true distribution over valuations belongs to a constrained Legendre polynomial sieve space. Bierens (2008) shows the Legendre polynomials span the set of distributions over the unit interval and also presents a very weak constraint on the smoothness of the distribution over valuations that ensures compactness. Adopting the Legendre polynomials as the sieve space requires embedding the support for the distribution over valuations within the unit interval, which Bierens (2008) and Bierens and Song (2010) accomplish using the Gaussian CDF. For computational purposes here, rather than using a non-linear transformation of the valuations that allows for unbounded support, the following

assumption embeds a bounded set of possible valuations in the unit interval with a linear transformation:

Assumption 4 (A Priori Bounded Support) *There exists some upper bound for valuations $\bar{M} > \bar{x} > 0$ known to the econometrician a priori, allowing the support for valuations to be mapped into the unit interval using the linear transformation:*

$$G(x) = \frac{x}{\bar{M}} \quad g(x) = \frac{1}{\bar{M}}$$

The asymptotic analysis in the consistency argument is complicated by an upper semicontinuous population log-likelihood criterion function, which arises due to different bidder-types bidding over different supports. These discontinuities are such that neither the USLLN in Bierens (2008) nor Bierens and Song (2010)'s generalization of Jennrich's USLLN, which accommodates discontinuities in the sample criterion function, apply. To address this irregularity in the criterion function, I extend the parametric consistency result for upper semicontinuous population criterion functions from Chernozhukov and Hong (2004) to the infinite dimensional case. This extension adopts the USLLN from Artstein and Wets (1995), who use the same notion of weak epi-convergence as in Hess (1996) and Chernozhukov and Hong (2004) to establish uniform convergence of functions subject to a local uniform integrability condition.

Since the valuations are bounded above and all players follow continuously differentiable, strictly monotonic strategies, there are no atoms in the distribution over bids and the criterion function satisfies a local uniform bound that suffices for applying the Artstein and Wets (1995) uniform strong law of large numbers. Convergence of the criterion function at its optimum follows from the USLLN and continuity of the population criterion function at its optimum. Lastly, Bierens and Song (2010) show that the criteria function's optimum over a sequence of constrained sieve spaces converges to the the global optimum of the unconstrained sieve space under very general conditions on the truncation algorithm. These arguments suffice to establish consistency.

Theorem 5 (Consistency of SNP-ML Estimator)

Suppose Assumptions 1 - 4 hold, Theorems 1, 2, or 4 apply so that the level-k auction

model is identified, and θ_0 belongs to the constrained Legendre polynomial sieve space endowed with metric ρ . Let n_N be an arbitrary subsequence of n such that $\lim_{N \rightarrow \infty} n_N = \infty$ and $\lim_{N \rightarrow \infty} \frac{n}{N} = 0$. Then for the maximum likelihood estimator $\tilde{\theta}_N$, $\rho(\tilde{\theta}_N, \theta_0) \rightarrow 0$ a.s.

The experimental ideal for estimating the auction model would be to observe bidding behavior in a series of identical auctions. While these settings do not obtain in empirical work, a common practice uses observable auction-specific covariates to control for auction-specific heterogeneity. In fact, the link between observable features of the object for sale and the distribution over valuations is frequently the primary concern of the empirical exercise.⁸ For additional tractability, I assume a separable structure for individual valuations so that:

Assumption 5 (Separable Auction-Specific Heterogeneity) *Given a set of auction-specific covariates, Z_t , bidder i 's valuation for the object at auction, X_{it} is given by:*

$$\log(X_{it}) = \gamma'Z_t + U_{it}$$

where $U_{it}|Z_t \perp\!\!\!\perp U_{jt}|Z_t, \forall i \neq j$.

Non-separable auction-specific heterogeneity can be allowed at the expense of substantial complexity, and so assumption 5 is made for numerical and notational parsimony that allows simply appending γ to the parameter vector θ .

In implementation, computing the level- k auction model presents two key numerical challenges. First, simply computing the likelihood requires solving non-linear bidding functions and inverse bidding functions that preclude analytical solutions.⁹ Second, the mixture-of-types structure in the likelihood requires an expectation-maximization (EM) algorithm for

⁸Note that I do not address unobserved heterogeneity across auctions here. A great deal of research had focused on developing strategies for addressing auction-specific heterogeneity in estimating auctions. A number of works, including Bajari and Ye (2003) and Hong and Shum (2002) link unobserved heterogeneity to the number of bidders in the auction. Haile, Hong, and Shum (2003) use multiple bids observed in each auction to control for auction specific heterogeneity, a strategy further developed by Krasnokutskaya (Forthcoming - 2009), An, Hu, and Shum (Forthcoming), and Hu, McAdams, and Shum (2009). It is likely that a similar technique could be adapted to level- k auctions but this analysis is beyond the scope the current work.

⁹This computational burden has often lead researchers to adopt a nonlinear least squares or simulated method of moments approach to estimation, but these techniques cannot be directly applied here due to the failure of revenue equivalence. Laffont, Ossard, and Vuong (1995) presented seminal contributions in this line of research with a host of additional estimation strategies surveyed in Paarsch and Hong (2006).

finding the maximum, which is computationally infeasible given the challenges associated with maximizing the expected likelihood. To address this complication, a generalized expectation maximization algorithm partitions parameter space into a set of parameters for which closed-form solutions are readily available and another set requiring numerical methods for optimization. Partially maximizing over those parameters for which closed-form solutions are available before maximizing over the full set of parameters greatly reduces computation time, rendering calculation of the estimator feasible. I discuss this algorithm and related implementation issues for using the Legendre polynomials further in the technical web appendix.

4.1 Monte Carlo Simulations

Two Monte Carlo Simulation exercises evaluate the performance of maximum likelihood estimation in the level- k model. The first exercise draws valuations from a log-normal distribution with mean parameter, $\mu = 0$ and standard error parameter $\sigma = 0.5$ truncated to ensure bounded support. Estimating the level- k auction model uses a correctly-specified parametric model having observed 60, 120, or 300 bids from simulated bidders competing in auctions with $N \in \{3, 4, 5, 6, 12\}$ for total sample sizes of 300, 600, and 1,500 bids. The second exercise draws valuations from a Legendre Sieve distribution with unit support and parameter vector $\theta = (-0.25, -0.05)$ to test the sieve estimator in a properly specified model with $N \in \{2, 3, 4\}$ for a total sample of 600 bids. Both simulations implement a model with three behavioral types: Equilibrium, Random Level-1, and Random Level-2 representing 20%, 60%, and 20% of the population, respectively.

Table 1 presents results from a set of 100 simulations under the log-normal specification. The estimator retains consistency with the MSE diminishing at roughly the expected rate as the number of observations increases. However, it is worth noting that, while estimates are very precise relating to the parameters governing the distribution over valuations, the estimates for the distribution over types are still quite noisy, maintaining a standard deviation around 10% even with 1,500 observations. The convergence properties of the Legendre sieve estimator are quite similar.

These simulations indicate that, while estimates for the distribution over valuations have

Table 1: Monte Carlo Simulations for Parametric Estimation

	P_{Eqm}	P_{L1R}	P_{L2R}	Mean	StDev
Population	0.2	0.6	0.2	0	0.5
Log Normal Parametric Estimator					
# of Obs	Mean Square Error (*100)				
300	3.500	2.820	3.940	0.020	0.030
600	2.240	1.030	2.850	0.010	0.010
1,500	2.110	0.580	2.040	-	-
# of Obs	Standard Deviation				
300	0.185	0.168	0.197	0.013	0.018
600	0.132	0.101	0.159	0.008	0.012
1,500	0.109	0.074	0.120	0.005	0.007
Legendre Sieve Estimator					
# of Obs	Mean Square Error (*100)				
600	2.417	0.615	2.449	0.049	0.050
	Standard Deviation				
600	0.155	0.078	0.155	0.022	0.022

This table reports the result of maximum likelihood estimation for a parametric simulation where individual valuations are drawn from a truncated Lognormal distribution and individual bids are chosen according to a randomly assigned behavioral type. These estimator results are generated from 100 simulated samples and illustrate both the consistency of the estimation strategy and the need for a robust sample size for precise estimation. Note: 37.9% of simulations resulted in corner solutions for P_{Eqm} or P_{L2R} .

reasonable accuracy and precision, it is difficult to get precise findings differentiating between models of bidders that are more sophisticated than the $L1R$ bidder type. The imprecise estimation highlights the importance of working with a partially identified model, motivating the focus in the next section on developing applied mechanism design strategies that are robust to unidentified distributions over bidder-types. Also, the weak separation among higher-order types indicates the most relevant empirical distinction is between sophisticated bidding behavior that accounts for others' bid shading and unsophisticated bidding behavior responding to an uninformative model of others.

5 Expected Revenues & Optimal Reserve Pricing

One of the primary applications in analyzing auction data is to facilitate mechanism design decisions such as optimally setting the reserve price or choosing between a first and second price auction. Crawford, Kugler, Neeman, and Pauzner (2009) extend the Crawford and Iriberri (2007a) model to auctions with a reserve price. Their analysis illustrates the effects of behavioral agents on optimal auction design with representative examples by focusing on simple settings with two bidders when the mechanism designer knows these bidders' types. In the technical web appendix, I develop this analysis further, characterizing expected revenues in auctions with more than two bidders under a general distribution over valuations where the composition of bidder-types in the population is unknown and may be unidentified.¹⁰

Given the analysis in Crawford, Kugler, Neeman, and Pauzner (2009) and in the technical web appendix defining the behavioral model, we can simply incorporate the reserve price, denoted r , as an additional parameter in the distribution for the value of the winning bid as $f_W(w; p, \theta, r)$. Then, the expected utility to a seller who attaches the value v_s to the object at auction is given by:

$$E[U_s|r] = v_s F_X(r; \theta)^N + \int_r^{\bar{x}} w f_W(w; p, \theta, r) dw \quad (4)$$

Solving the first order conditions for maximizing the seller's welfare following Myerson (1981)'s analysis gives an optimal reserve price as the solution to a fixed point problem:

$$r = v_s \frac{N F_X(r; \theta)^{N-1} f_X(r; \theta)}{f_W(r; p, \theta, r)} + \frac{\int_r^{\bar{x}} w \frac{\delta f_W(w; p, \theta, r)}{\delta r} dw}{f_W(r; p, \theta, r)} \quad (5)$$

¹⁰The application here simply identifies the optimal reserve price in a first-price auction, which is quite narrow in scope relative to designing the revenue-maximizing mechanism. Since revenue equivalence fails in the presence of level- k bidders, Myerson (1981)'s optimal auction result does not apply so there is no reason to expect a first-price auction with reserve prices to be an optimal mechanism. To illustrate this, Crawford, Kugler, Neeman, and Pauzner (2009) present an exotic mechanism that generates greater expected revenues than is attainable in the first price auction with reserve price. Focusing on a relatively simple deviation from the original mechanism used to estimate the model provides greater confidence in the counterfactual analysis, in particular regarding bidders' response to changes in the reserve price, as we implicitly assume that an individuals' type will be independent of the auction format. Implementing dramatic changes in the structure of bidding and allocation rules could affect players' participation decisions as well as their position in the behavioral hierarchy, invalidating this assumption. Further, while it is fairly direct to compare reserve prices under different mixtures of behavioral types, it is difficult to draw conclusions comparing the optimal auctions belonging to a larger set of possible mechanisms.

Equation 5 provides an implicit solution for the optimal reserve price given the distribution over valuations and bidder-types. In practice, simulation methods to choose the optimal reserve price to maximize the conditional expected revenues in equation 4 work well here.

When variation in the number of bidders is not exogenous, the exclusion restrictions establishing identification no longer apply. In this setting, the distribution over types is unidentified and the expected revenue at a given reserve price is partially identified, belonging to a compact, convex set and the optimal reserve price belongs to a compact identified set. To show this result, note that the right hand side of equation 5 is continuous in changes to the distribution over bidder-types and bounded away from zero due to the regularity conditions that ensure continuously differentiable bidding strategies for all bidder-types. First, as can be verified by observing that all elements in the population likelihood for the winning bid are continuous polynomials in p_k , $f_W(r; p, \theta, r)$ is continuous in the distribution over bidder-types. Similarly, as is shown in Crawford, Kugler, Neeman, and Pauzner (2009), each of the level- k bidder-types behavioral strategies are continuous in the reserve price. As such, $\frac{\delta f_W(w; \theta, r)}{\delta r}$ is continuous in p_k . Finally, searching over the set of bidder-types completely characterizes the identified set.

Theorem 6 (Partially Identified Expected Revenue & Optimal Reserve Price)

Suppose the distribution over bidder-types is not identified in the level- k auction model. Then:

1. *The seller's expected revenue at a given reserve price belongs to a closed, convex identified set that is bounded above and below by the expected revenues generated by the homogeneous population of bidder-types that maximize and minimize expected revenues.*
2. *The optimal reserve price from equation 5 belongs to a partially identified compact set.*

6 Optimal Reserve Pricing in USFS Timber Auctions

With well established publicly available data, timber auctions sponsored by the US Forestry Service receive a great deal of attention in the literature on empirical methods for optimal mechanism design.¹¹ As such, this setting provides an ideal environment to compare the

¹¹I use data provided by Philip Haile that has been analyzed extensively in empirical studies of auctions. Early studies looking at this data include Baldwin, Marshall, and Richard (1997), who provide a detailed

Table 2: Summary Statistics for USFS Timber Auction Data

# of Bidders	# of Auctions	# of Obs	Mean Appraisal (\$000)	Mean Bid (\$000)	St Dev Bid (\$000)
2	143	286	886	1,215	1,690
3	75	225	810	1,276	1,512
4	68	272	1,401	2,256	3,107
5	57	285	2,480	3,861	6,912
6	26	156	1,719	2,897	3,101
7	23	161	2,518	5,112	9,325
8	6	48	701	1,188	621
9	9	81	3,245	6,585	5,907
10	13	130	4,865	11,748	19,871
Full Sample		1,655	1,892	3,481	7,780

This table reports summary statistics for USFS Timber Auction Data. The analysis uses only the auctions with 2-4 bidders, which still leaves a sample of over 744 bids for estimation.

mechanism design implications of a level- k behavioral model for IPV first price auctions with the equilibrium results. The data treatment is based on the results from Haile, Hong, and Shum (2003), whose findings support the IPV model for sealed-bid timber auctions of scaled sale contracts. In these contracts, logging companies pay a price for timber harvesting rights based on the actual timber harvested, greatly reducing common- and affiliated-value components in determining the individual firm's valuation. The sample focuses on sealed bid sales from 1982-1996 that had between two and four bidders, excluding salvage sales, tracts set aside for sale to small businesses, and auctions that had more than 4 bidders. Bids from auctions in the highest and lowest 1% quantiles of appraised values are trimmed from the sample, though this had little impact on the results. For completeness, Table 2 presents summary statistics characterizing the entire sample of bids, though only 744 of these observations are selected after trimming.

Figure 6 presents the estimated distributions over valuations under homogeneous bidder-

institutional analysis of the auctions in testing for collusion among bidders. More recent studies in empirical industrial organization by Athey and Levin (2001), Athey, Levin, and Seira (2008), and Haile and Tamer (2003) have focused on mechanism design issues in USFS timber auctions. Haile (2001) looks at the role of resale in affecting valuations for timber auctions and Haile, Hong, and Shum (2003) use USFS data to test for common value components in bidder valuations. Campo, Guerre, Perrigne, and Vuong (Forthcoming) and Lu and Perrigne (2008) use USFS auction data to characterize risk aversion within the bidding population. Aradillas-Lopez, Gandhi, and Quint (2010) develop tests for the hypothesis that bidder participation varies exogenously, with empirical results verifying the exogenous bidder participation in USFS timber auctions.

type specifications. Panels A and B present the estimation results for the level- k bidder-types estimated using several different specifications for the polynomial order of the SNP-ML Estimator. The patterns across estimation models are largely as expected. The $L1_R$ bidder-type's distribution over valuations in Panel A is scaled and left-shifted relative to the distribution over bids, with the SNP-ML estimator converging quickly to a kernel based estimator for the $L1_R$ bidder-type derived in the web appendix. The Equilibrium bidder-type's distribution over valuations in Panel C is estimated using the Guerre et al. (2000) estimator, with a substantially fatter tail than either of the distributions recovered from a model that assumes Level- k bidder-types. This fatter tail is consistent with the implication that less sophisticated bidders over-bid relative to equilibrium bidder-types.

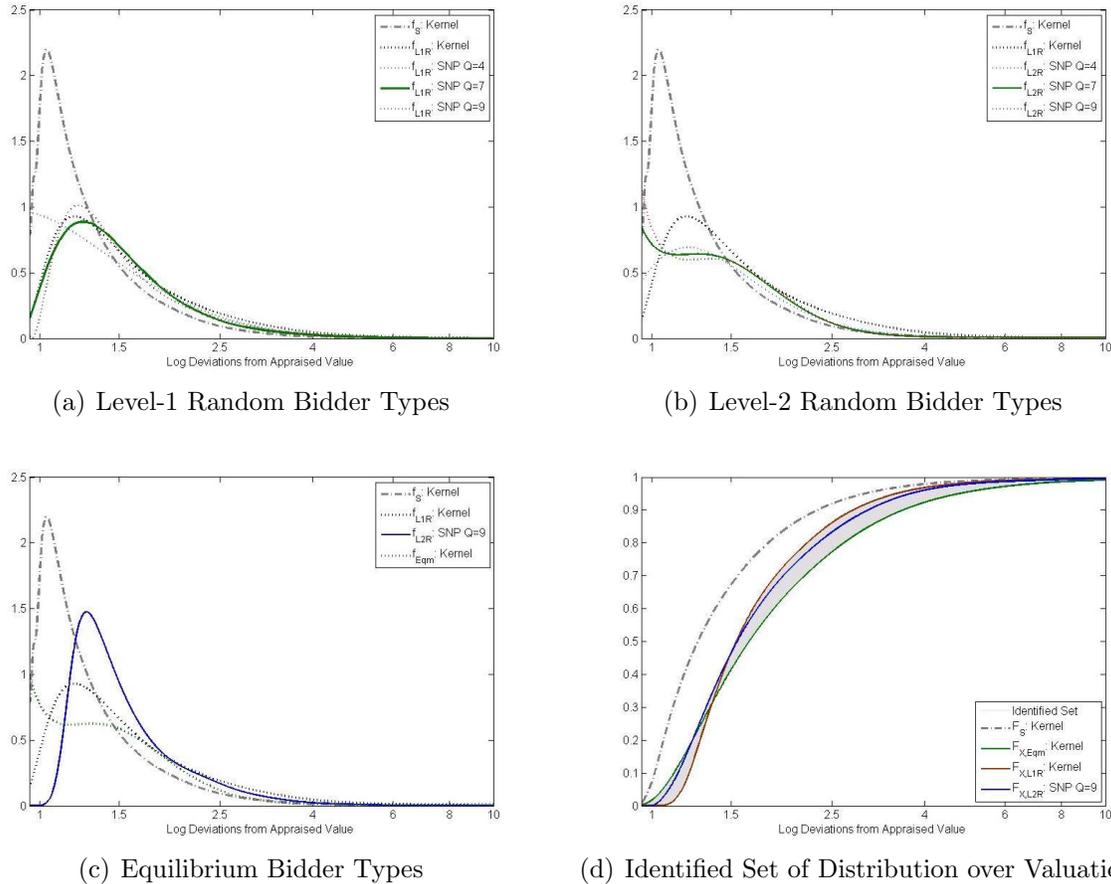


Figure 6: Distributions for Valuations in USFS Timber Auctions

Panels (A) - (C) represent the distribution over valuations implied by observed bidding behavior in USFS timber auctions when the population is comprised entirely of level-1 random, level-2 random, or Equilibrium bidder-types, respectively. Panel D illustrates the identified set of cumulative densities for the true distribution over bids.

The $L2_R$ bidder-type’s implied distribution over valuations illustrates an interesting feature related to the potential for overfitting the model that may not be captured directly through the likelihood ratio. In particular, the $L2_R$ bidder model fits the data by making the derivative of the bidding function as small as possible near the mode of the distribution over valuations, creating a spike in the likelihood. This spike is mitigated by the assumption that bidding behavior is strictly monotonic, as formalized in Lemma 1, though explicitly incorporating this restriction into the estimation procedure is not entirely trivial. Henderson et al. (2008) look at ways to enforce monotonicity in estimation for auctions under the equilibrium behavioral model using kernel density estimation methods. While their techniques are not directly transferable to sieve estimation, it may be possible to use a constrained sieve estimator or penalized likelihood criterion function.

The last panel bounds the identified set implied by the observed distribution over bids. This panel illustrates the result that, even when the distribution over valuations is not identified, the bounds on the distribution over valuations alone are still rather informative. This finding is consistent with simulation evidence that the ability to empirically separate types is relatively weak while not necessarily hindering meaningful inference on the primitives defining valuations in the auction model.

Turning towards the optimal reserve price begins by briefly addressing the specification of the truncation of the SNP-ML. Model selection statistics for the SNP-ML estimator of the distribution over bids in homogeneous populations are presented in Table 3. Likelihood statistics generally favor the Random Level-2 bidder model with a very flexible distribution over bids. However, the visual evidence of over-fitting for this model is too great to ignore, so the analysis for expected revenues and optimal reproceeds with the distribution over valuations estimated from the Random Level-2 bidder-type truncating to a 5th order Legendre polynomial. Treating this estimated distribution as if it were the true distribution over valuations, the analysis turns to consider inference for expected revenues and optimal reserve prices in an incompletely identified model.

Figure 7 presents counterfactual analysis of the effect of changing the reserve price for a population of $N = 4$ bidders. First, note that the bidding strategies separate as expected, with more aggressive bid shading by the higher-level types. The Random Level-1 bidder type

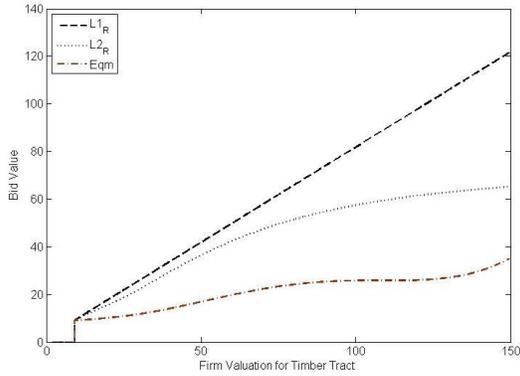
Table 3: Model Selection Statistics for USFS Timber Auction Data

Polynomial Order	Level 1 Random		Level 2 Random	
	Log Likelihood	BIC	Log Likelihood	BIC
3	(10,324)	20,674	(10,148)	20,322
4	(10,309)	20,652	(9,979)	19,990
5	(10,298)	20,635	(9,775)	19,589
6	(10,295)	20,636	(9,774)	19,594
7	(10,291)	20,635	(9,740)	19,533
8	(10,286)	20,632	(9,660)	19,379
9	(10,284)	20,634	(9,652)	19,369
10	(10,283)	20,638	(9,629)	19,330

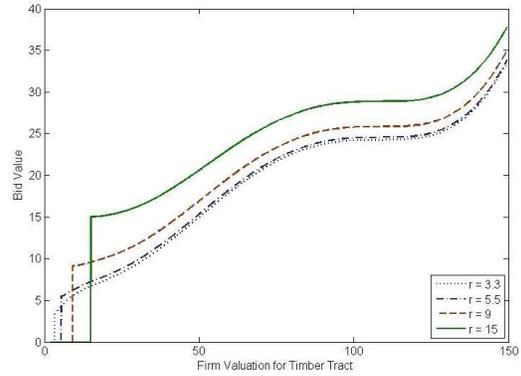
This table reports model selection statistics for USFS Timber Auction Data. These results are generated from observed bids in 744 auctions and provide substantial support to the hypothesis that bidders in this setting are sophisticate, though the BIC selected model appears to substantially overfit the data.

is particularly insensitive to the reserve price, which is consistent with that bidder-type’s insensitivity to the distribution over valuations in choosing their bid shade. The Level-2 and Equilibrium bidder-types show significant strategic responses to the reserve price, as they condition on the event that entrants to the auction will have valuations exceeding the reserve price.

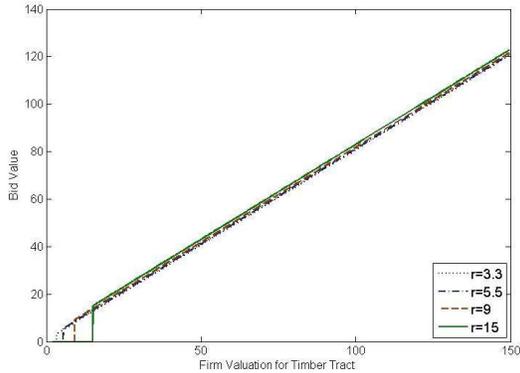
The effect of the reserve price on expected revenue from the auction is depicted in Figure 8, which plots the expected revenue from the auction at various levels of the reserve price. As is evident in Figure 8, the identified set for the optimal reserve price is quite large, driven mainly by the Random Level-1 bidder’s lack of sensitivity to the reserve price. Indeed, the optimal reserve price in an auction with a population entirely formed of Level-1 Random bidder-types would be equal to the seller’s own valuation for the good. Note that the equilibrium optimal reserve price quite far out in the tail, resulting in a nearly 70% chance that the auction will close without a buyer. This feature helps to rationalize the fact that observed reserve prices are often non-binding or otherwise seem “too low” in empirical analysis, as less aggressive reserve pricing could reflect skepticism on the part of the seller that higher-order bidder types dominate the population. For this reason, previous researchers have argued that the non-binding reserve price as driven by non-revenue motives related to forest management and resource development. The analysis here implies uncertainty regarding the



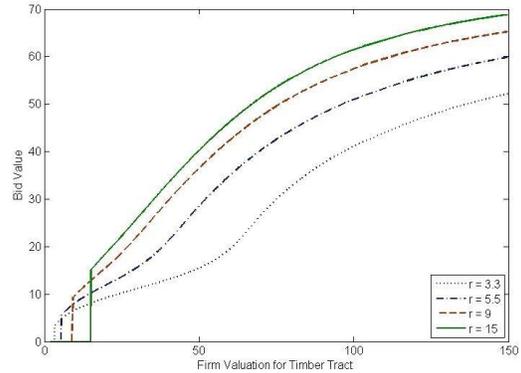
(a) All Bidder-type Strategies



(b) Equilibrium Bidder Types



(c) Random Level-1 Bidder-Type



(d) Random Level-2 Bidder-Type

Figure 7: Implied Bid Function Responses to Reserve Prices

The bidding functions implied under alternate reservation prices for USFS timber auctions for the random level-1 bidder-type are much less sensitive than the rules followed by more sophisticated bidders. As such, the potential benefit of shifting the reservation price in a less sophisticated population is not as great as for higher order types.

bidders' strategic response to the reserve price also rationalizes this policy.

7 Conclusion

This paper proposes a structural econometric model for analyzing auction data when bidder behavior is governed by a level- k behavioral model, establishing identification conditions for the model and developing a nonparametric consistent estimation strategy. I apply the model to the applied mechanism design problem of finding the optimal reserve price in first price auctions with heterogeneous non-equilibrium behavior with findings that underscore the degree to which behavioral misspecification can affect counterfactual analysis. The important

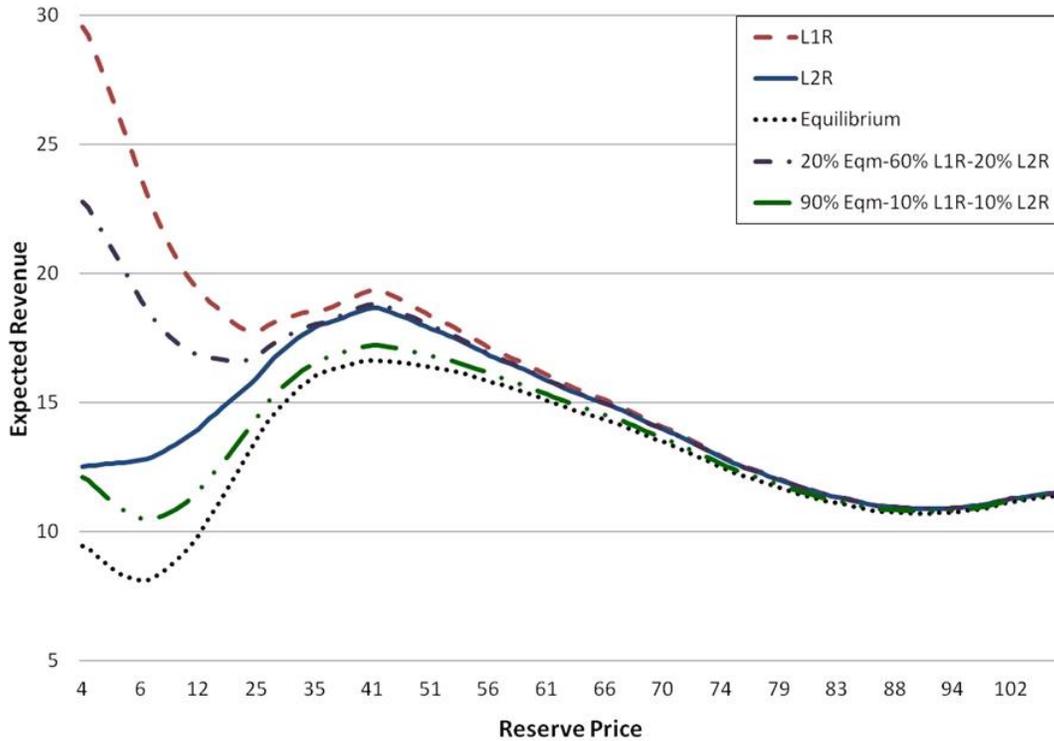


Figure 8: Expected Revenues for USFS Timber Auctions

This figure plots the effect of the reserve price on expected revenues in USFS Timber Auctions. When the population is dominated by the Level-1 Bidder-type, the seller maximizes revenue by setting the reserve price equal to their value for the good. When the seller faces a population with greater sophistication, the strategic benefits of the reserve price outweigh the costs from potentially losing a sale, yielding a relatively high optimal reserve price.

Lessons to draw from the analysis is that, while it is possible to attain identification in the face of strategic heterogeneity, it may be quite difficult to exploit that identification in practice. Further, accounting for potential strategic uncertainty can have significant effects on design recommendations and expected revenues.

An important open problem in this research is the need to address statistical decision making in incompletely identified and partially identified models. Decision theorists have made considerable progress in providing foundations for ambiguity averse preferences and some work has been done on inference in this context using min-max preferences.¹² These

¹²The axiomatic choice framework introduced by Gilboa and Schmeidler (1989) rationalizes a robust decision rule that maximizes expected utility generated from the state in the identified set that minimizes expected utility conditional on the chosen action. Axiomatic treatments of decision in the presence of ambiguity date to Savage (1954), coming into stark focus with the Ellsberg (1961) paradox. Recent advances, in this area, including Klibanoff et al. (2005), Maccheroni, Marinacci, and Rustichini (2006), and Klibanoff,

preferences are readily applied to estimating the ambiguity-robust optimal reserve price in incompletely-identified models since, for a given reserve price, the distribution over bidder-types generating the minimum expected utility to the seller will be degenerate, placing all mass on the single bidder-type that minimizes the seller's expected revenue at that reserve price. This regularity feature greatly facilitates analyzing optimal reserve pricing in the current context and is readily extended to other auction settings, such as incorporating unknown risk aversion.

One possible extension of these results could address heterogeneity in both strategic beliefs and risk aversion by combining information from multiple auction mechanisms. For example, Lu and Perrigne (2008) leverage a second-price auction where an individual's decision is free of strategic and risk considerations to identify the distribution over valuations and a first-price auction to identify the bidder's utility functions. Another approach might be to analyze a model similar to Campo, Guerre, Perrigne, and Vuong (Forthcoming) and exploiting random variation in the reserve price to pin down sufficient quantiles of the distribution over bids to identify the distribution over bidder-types. Li (2005) and Li and Perrigne (2003) consider the identification problem with random reserve prices, though since the reserve prices are hidden, Li (2005) and Li and Perrigne (2003) show the uncertainty introduced to the bidding problem complicates identification rather than generating additional information on bidding characteristics in the population.

Adopting non-equilibrium behavioral economic models for structural econometric analysis could yield interesting insights in other strategic environments, such as the estimation of static and dynamic games often studied in empirical industrial organization. One such application could develop a level- k analog to the cognitive hierarchy model in Goldfarb and Xiao (2010) for static entry games in markets, such as those pioneered by Bresnahan and Reiss (1991) and Berry (1992). Ciliberto and Tamer (Forthcoming 2009) analyze this problem for airlines when the equilibrium is partially identified, finding that the equilibrium prediction underestimated coordination among airlines in reaction to a change in regulatory

Marinacci, and Mukerji (2009) have extended Gilboa and Schmeidler (1989)'s results to allow for smooth preferences over ambiguity. Recent applications of these techniques to empirical decision problems include: Garlappi, Uppal, and Wang (2007) in portfolio selection, Manski (2007) and Stoye (2007) in treatment selection, and Handel, Misra, and Roberts (2009) in optimal pricing.

policy. Several authors, including Rapoport, Seale, and Winter (2002) and Camerer, Ho, and Chong (2004) show that players in the lab often achieve better ex-post coordination than equilibrium predicts and that this coordination is consistent with a cognitive hierarchy model. Using the Aradillas-Lopez and Tamer (2008) approach to estimating games based on rationalizability assumptions, akin to the analysis in Collard-Wexler (2008) or in a dynamic context following Aguirregabiria and Magesan (2009) provide two interesting potential methods for relaxing the equilibrium assumption in this context.

References

- Victor Aguirregabiria and Arvind Magesan. Estimation of dynamic discrete games when players beliefs are not in equilibrium. *Mimeo*, 2009.
- Yonghong An, Yingyao Hu, and Matthew Shum. Estimating first-price auction models with unknown number of bidders: a misclassification approach. *Journal of Econometrics*, Forthcoming.
- Andres Aradillas-Lopez and Elie Tamer. The identification power of equilibrium in simple games. *Journal of Business & Economic Statistics*, 26(3):261–283, JUL 2008. ISSN 0735-0015.
- Andres Aradillas-Lopez, Ahmit Gandhi, and Daniel Quint. Identification and testing in ascending auctions with unobserved heterogeneity. *Mimeo*, 2010.
- Zvi Artstein and Roger J.-B. Wets. Consistency of minimizers and the sln for stochastic programs. *Journal of Convex Analysis*, 2(1-2):1–17, 1995.
- S Athey and PA Haile. Identification of standard auction models. *Econometrica*, 70(6): 2107–2140, NOV 2002. ISSN 0012-9682.
- S Athey and J Levin. Information and competition in us forest service timber auctions. *Journal of Political Economy*, 109, 2001.
- Susan Athey and Philip A. Haile. *Handbook of Econometrics*, chapter Nonparametric Approaches to Auctions. Elsevier, 2005.
- Susan Athey, Jonathan Levin, and Enrique Seira. Comparing open and sealed bid auctions: Evidence from timber auctions. *NBER Working Papers*, 14590, 2008.
- P Bajari and A Hortacsu. Are structural estimates of auction models reasonable? Evidence from experimental data. *Journal of Political Economy*, 113(4):703–741, AUG 2005. ISSN 0022-3808.
- P Bajari and LX Ye. Deciding between competition and collusion. *Review of Economics and Statistics*, 85(4):971–989, NOV 2003. ISSN 0034-6535.
- LH Baldwin, RC Marshall, and JF Richard. Bidder collusion at forest service timber sales. *Journal of Political Economy*, 105, 1997.
- P Battigalli and M Siniscalchi. Rationalizable bidding in first-price auctions. *Games and Economic Behavior*, 45(1):38–72, OCT 2003. ISSN 0899-8256.
- Arie Beresteanu and Francesca Molinari. Asymptotic properties for a class of partially identified models. *Econometrica*, 76(4):763–814, JUL 2008. ISSN 0012-9682.
- BD Bernheim. Rationalizable Strategic Behavior. *Econometrica*, 52(4):1007–1028, 1984. ISSN 0012-9682.

- ST Berry. Estimation of a Model of Entry in the Airline Industry. *Econometrica*, 60(4): 889–917, JUL 1992. ISSN 0012-9682.
- Herman J. Bierens. Semi-nonparametric interval-censored mixed proportional hazard models: Identification and consistency results. *Econometric Theory*, 24(2003-09):749–794, September 2008.
- Herman J. Bierens and Hosin Song. Semi-nonparametric estimation of independently and identically repeated first-price auctions via simulated method of moments. *Mimeo*, (2003-09), September 2010.
- JC Biscarat. Almost Sure Convergence of a Class of Stochastic Algorithms. *Stochastic Processes and Their Applications*, 50(1):83–99, MAR 1994. ISSN 0304-4149.
- A Bosch-Domenech, JG Montalvo, R Nagel, and A Satorra. One, two, (three), infinity, ... : Newspaper and lab beauty-contest experiments. *American Economic Review*, 92(5): 1687–1701, DEC 2002. ISSN 0002-8282.
- Bjarne Brendstrup and Harry J. Paarsch. Identification and estimation in sequential, asymmetric, english auctions. *Journal of Econometrics*, 134(1):69–94, September 2006.
- TF Bresnahan and PC Reiss. Entry and Competition in Concentrated Markets. *Journal of Political Economy*, 99(5):977–1009, OCT 1991. ISSN 0022-3808.
- CF Camerer, TH Ho, and JK Chong. A cognitive hierarchy model of games. *Quarterly Journal of Economics*, 119(3):861–898, AUG 2004. ISSN 0033-5533.
- Sandra Campo, Emmanuel Guerre, Isabelle Perrigne, and Quang Vuong. Semiparametric estimation of first-price auctions with risk averse bidders. *Review of Economic Studies*, Forthcoming.
- Joe Casella and Roger Berger. *Statistical Inference*. Duxbury Press, 2001.
- KS Chan and J Ledolter. Monte-Carlo EM Estimation for Time-Series Models Involving Counts. *Journal of the American Statistical Association*, 90(429):242–252, MAR 1995. ISSN 0162-1459.
- Xiaohong Chen. Large sample sieve estimation of semi-nonparametric models. *Handbook of Econometrics*, 6B:5549–5632, 2007.
- V Chernozhukov and A Hong. Likelihood estimation and inference in a class of nonregular econometric models. *Econometrica*, 72(5):1445–1480, SEP 2004. ISSN 0012-9682.
- Victor Chernozhukov, Han Hong, and Elie Tamer. Estimation and confidence regions for parameter sets in econometric models. *Econometrica*, 75(5):1243–1284, SEP 2007. ISSN 0012-9682.
- Federico Ciliberto and Elie T. Tamer. Market structure and multiple equilibria in the airline industry. *Econometrica*, Forthcoming 2009.

- Allan Collard-Wexler. Discussion of aradillas-lopez and tamer. *Journal of Business and Economic Statistics*, 2008.
- M Costa-Gomes, VP Crawford, and B Broseta. Cognition and behavior in normal-form games: An experimental study. *Econometrica*, 69(5):1193–1235, SEP 2001. ISSN 0012-9682.
- Miguel A. Costa-Gomes and Vincent P. Crawford. Cognition and behavior in two-person guessing games: An experimental study. *American Economic Review*, 96(5):1737–1768, DEC 2006. ISSN 0002-8282.
- V. Crawford, T. Kugler, Z. Neeman, and A. Pauzner. Behaviorally optimal auction design: An example and some observations. *Journal of the European Economic Association*, 7, 2009.
- Vincent Crawford and Nagore Iriberri. Level-k auctions: Can a nonequilibrium model of strategic thinking explain the winner’s curse and overbidding in private-value auctions? *Econometrica*, 75(6):1721–1770, NOV 2007a. ISSN 0012-9682.
- Vincent P. Crawford and Nagore Iriberri. Fatal attraction: Salience, naivete, and sophistication in experimental “Hide-and-Seek” games. *American Economic Review*, 97(5):1731–1750, DEC 2007b. ISSN 0002-8282.
- VP Crawford. Lying for strategic advantage: Rational and boundedly rational misrepresentation of intentions. *American Economic Review*, 93(1):133–149, MAR 2003. ISSN 0002-8282.
- SG Donald and HJ Paarsch. Identification, estimation, and testing in parametric empirical models of auctions within the independent private values paradigm. *Econometric Theory*, 12(3):517–567, AUG 1996. ISSN 0266-4666.
- MA El Gamal and DM Grether. Are people Bayesian? Uncovering behavioral strategies. *Journal of the American Statistical Association*, 90(432):1137–1145, DEC 1995. ISSN 0162-1459.
- D Ellsberg. Risk, Ambiguity, and the Savage Axioms. *Quarterly Journal of Economics*, 75(4):643–669, 1961. ISSN 0033-5533.
- E Eyster and M Rabin. Cursed equilibrium. *Econometrica*, 73(5):1623–1672, SEP 2005. ISSN 0012-9682.
- Philippe Février. Nonparametric identification and estimation of a class of common value auction models. *Journal of Applied Econometrics*, 23(7):949–964, NOV-DEC 2008. ISSN 0883-7252.
- A. Ronald Gallant and Douglas W. Nychka. Semi-nonparametric maximum likelihood estimation. *Econometrica*, 55:363–390, 1987.

- Lorenzo Garlappi, Raman Uppal, and Tan Wang. Portfolio selection with parameter and model uncertainty: A multi-prior approach. *Review of Financial Studies*, 20(1):41–81, JAN 2007. ISSN 0893-9454. doi: {10.1093/rfs/hhl003}.
- I Gilboa and D Schmeidler. Max-min expected utility with non-unique prior. *Journal of Mathematical Economics*, 18, 1989.
- JK Goeree, CA Holt, and TR Palfrey. Quantal response equilibrium and overbidding in private-value auctions. *Journal of Economic Theory*, 104(1):247–272, MAY 2002. ISSN 0022-0531.
- JK Goeree, CA Holt, and TR Palfrey. Regular Quantal Response Equilibrium. *Experimental Economics*, 8(4):347–367, DEC 2005. ISSN 1386-4157.
- Avi Goldfarb and Mo Xiao. Who thinks about the competition? managerial ability and strategic entry in us local telephone markets. *Mimeo*, 2010.
- E Guerre, I Perrigne, and Q Vuong. Optimal nonparametric estimation of first-price auctions. *Econometrica*, 68(3):525–574, MAY 2000. ISSN 0012-9682.
- Emmanuel Guerre, Isabelle Perrigne, and Quang Vuong. Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions. *Econometrica*, 77(4): 1193–1227, JUL 2009. ISSN 0012-9682.
- P Haile. Auctions with resale markets: An application to us forest service timber sales. *American Economic Review*, 91, 2001.
- PA Haile and E Tamer. Inference with an incomplete model of English auctions. *Journal of Political Economy*, 111(1):1–51, FEB 2003. ISSN 0022-3808.
- Philip A. Haile, Han Hong, and Matthew Shum. Nonparametric tests for common values at first-price sealed-bid auctions. *SSRN eLibrary*, 2003.
- Philip A. Haile, Ali Hortacsu, and Grigory Kosenok. On the empirical content of quantal response equilibrium. *American Economic Review*, 98(1):180–200, MAR 2008. ISSN 0002-8282.
- Ben Handel, Kanishka Misra, and James Roberts. Robust firm pricing with panel data. *Mimeo*, 2009.
- David W Harless and Colin Camerer. The predictive utility of generalized expected utility theories. *Econometrica*, 62:1251–1289, 1994.
- Daniel Henderson, John List, Daniel Millimet, Christopher Parmeter, and Michael Price. Imposing monotonicity nonparametrically in first-price auctions. *Mimeo*, 2008.
- C Hess. Epi-convergence of sequences of normal integrands and strong consistency of the maximum likelihood estimator. *Annals of Statistics*, 24(3):1298–1315, JUN 1996. ISSN 0090-5364.

- TH Ho, C Camerer, and K Weigelt. Iterated dominance and iterated best response in experimental “p-beauty contests”. *American Economic Review*, 88(4):947–969, SEP 1998. ISSN 0002-8282.
- H Hong and M Shum. Increasing competition and the winner’s curse: Evidence from procurement. *Review of Economic Studies*, 69(4):871–898, OCT 2002. ISSN 0034-6527.
- Yingyao Hu, David McAdams, and Matthew Shum. Nonparametric identification of auction models with non-separable unobserved heterogeneity. *Mimeo*, 2009.
- Asen Ivanov, Dan Levin, and Muriel Niederle. Can relaxation of beliefs rationalize the winners curse?: An experimental study. *Econometrica*, 78:1435-1452, 2010.
- Hiroaki Kaido. A dual approach to inference for partially identified econometric models. *Mimeo*, 2009.
- P Klibanoff, M Marinacci, and S Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, NOV 2005. ISSN 0012-9682.
- Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji. Recursive smooth ambiguity preferences. *Journal of Economic Theory*, 144(3):930–976, MAY 2009. ISSN 0022-0531.
- Elena Krasnokutskaya. Identification and estimation in procurement auctions under unobserved auction heterogeneity. *Review of Economic Studies*, Forthcoming - 2009.
- Vijay Krishna. *Auction Theory*. Academic Press, 2002.
- JJ Laffont and Q Vuong. Structural econometric analysis of descending auctions. *European Economic Review*, 37, 1993.
- JJ Laffont, H Ossard, and Q Vuong. Econometrics of first price auctions. *Econometrica*, 63, 1995.
- T Li. Econometrics of first-price auctions with entry and binding reservation prices. *Journal of Econometrics*, 126, 2005.
- T Li and I Perrigne. Timber sale auctions with random reserve prices. *Review of Economics and Statistics*, 85(1):189–200, FEB 2003. ISSN 0034-6535.
- Jingfeng Lu and Isabelle Perrigne. Estimating Risk Aversion from Ascending and Sealed-Bid Auctions: The Case of Timber Auction Data. *Journal of Applied Econometrics*, 23(7): 871–896, NOV-DEC 2008. ISSN 0883-7252.
- Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498, NOV 2006. ISSN 0012-9682.
- Charles F Manski. Minimax-regret treatment choice with missing outcome data. *Journal of Econometrics*, 139:105–115, 2007.

- E Maskin and J Riley. Optimal auctions with risk averse bidders. *Econometrica*, 52 (6), 1984.
- RD McKelvey and TR Palfrey. Quantal Response Equilibria for Normal-Form Games. *Games and Economic Behavior*, 10(1):6–38, JUL 1995. ISSN 0899-8256.
- RD McKelvey and TR Palfrey. Quantal Response Equilibria for Extensive Form Games. *Experimental Economics*, 10(1):6–38, JUL 1998. ISSN 0899-8256.
- PR Milgrom and RJ Weber. A Theory of Auctions and Competitive Bidding. *Econometrica*, 50(5):1089–1122, 1982. ISSN 0012-9682.
- RB Myerson. Optimal Auction Design. *Mathematics of Operations Research*, 6(1):58–73, 1981. ISSN 0364-765X.
- R Nagel. Unraveling in guessing games: An experimental study. *American Economic Review*, 85(5):1313–1326, DEC 1995. ISSN 0002-8282.
- R Paarsch and H Hong. *An Introduction to the Structural Econometrics of Auction Data*. The MIT Press, 2006.
- DG Pearce. Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52(4):1029–1050, 1984. ISSN 0012-9682.
- Isabelle Perrigne and Quang Vuong. Identification and estimation of Bidders’ risk aversion in first-price auctions. *American Economic Review*, 97(2):444–448, MAY 2007. ISSN 0002-8282.
- A Rapoport, DA Seale, and E Winter. Coordination and learning behavior in large groups with asymmetric players. *Games and Economic Behavior*, 39(1):111–136, APR 2002. ISSN 0899-8256.
- Brian W. Rogers, Thomas R. Palfrey, and Colin F. Camerer. Heterogeneous quantal response equilibrium and cognitive hierarchies. *Journal of Economic Theory*, 144(4):1440–1467, JUL 2009. ISSN 0022-0531.
- Leonard J Savage. *The Foundations of Statistics*. Dover Publications, 1954.
- Robert Sherman, Sid Dalal, and Yu-Yun Ho. Conditions for convergence of monte-carlo em sequences with an application to diffusion modeling. *Econometrics Journal*, 2:248–267, 1999.
- DO Stahl and PW Wilson. Experimental Evidence on Players Models of Other Players. *Journal of Economic Behavior & Organization*, 25(3):309–327, DEC 1994. ISSN 0167-2681.
- DO Stahl and PW Wilson. On Players Models of Other Players - Theory and Experimental Evidence. *Games and Economic Behavior*, 10(1):218–254, JUL 1995. ISSN 0899-8256.
- Jorg Stoye. Minimax regret treatment choice with incomplete data and many treatments. *Econometric Theory*, 23:190–199, 2007.

Appendix 1: Proofs

Proof of Theorem 1

To begin, note the identified set contains any distribution F^* consistent with the equality:

$$F_X(\sigma^{-1}(x)) = F_S(x) = F^*(\sigma_*^{-1}(x)) \quad (\text{A-1})$$

The proof that $F^* = F_X$ centers on assumption 4, which ensures an individual's bid is equal to their valuation minus a non-negative, continuously differentiable bid shade that is zero for valuations arbitrarily close to zero. These properties establish two contradictions to complete the proof.

First, define $\epsilon_1 \equiv \inf \{x : F_X(x) \neq F^*(x)\} > 0$ and $\epsilon_2 \equiv \inf \{x > \epsilon_1 : F_X(x) = F^*(x)\} > \epsilon_1$ so that for $y \in [0, \epsilon_1)$, $F_X(y) = F^*(y)$ and, as such, the inverse bidding functions are identical to one another in this region, i.e., $\sigma^{-1}(y) = \sigma_*^{-1}(y)$. Note that $\sigma^{-1}(y)$ is always greater than y , continuous, and strictly increasing, so there is some $\tilde{y} < \epsilon$ with $\sigma^{-1}(\tilde{y}) \in (\epsilon_1, \epsilon_2)$. Then, $F_X(\sigma^{-1}(\tilde{y})) \neq F^*(\sigma_*^{-1}(\tilde{y}))$, contradicting A-1. As such, any distribution satisfying condition A-1 other than F_X must differ from F_X starting at the origin.

Now, suppose the distributions F_X and F^* diverge immediately from the origin and, wlog, that $F_X(x) > F^*(x)$ for $x \in [0, \epsilon)$ where $\epsilon \equiv \sup \{x : F_X(x) > F^*(x)\}$. In this case, the condition in equation A-1 implies $\sigma_k^{-1}(x) < \sigma_{k,*}^{-1}(x)$. However, there must come a point in $[0, \epsilon]$ where the distribution F^* begins “catching up” with F_X , i.e., where $f_X(x) < f^*(x)$. Further, by iterating down the hierarchy of bidder-types, these two inequalities imply the Jacobian terms also satisfy: $\frac{d\sigma_{k-1}^{-1}(s)}{ds}|_{s=\sigma_k(x)} < \frac{d\sigma_{k-1,*}^{-1}(s)}{ds}|_{s=\sigma_{k,*}(x)}$. The three inequalities combined imply:

$$\frac{F_X(x)}{f_X(x) \frac{d\sigma_{k-1}^{-1}(s)}{ds}|_{s=\sigma_k(x)}} > \frac{F^*(x)}{f^*(x) \frac{d\sigma_{k-1,*}^{-1}(s)}{ds}|_{s=\sigma_{k,*}(x)}} \quad (\text{A-2})$$

This inequality requires the bid shade under the alternative distribution, F^* to be *greater* than the bid shade under the true distribution, contradicting $\sigma_k^{-1}(x) < \sigma_{k,*}^{-1}(x)$ and proving the result.

Proof of Theorem 3

First, suppose $K = 2$ with known bidding strategies $\sigma_1(x)$ and $\sigma_2(x)$, the first step is to separate the mixture distribution of bids into the distribution over bids for homogeneous populations and use these components to recover the distribution over valuations. Here, the mixture distribution over bids can be written as:

$$F_{S,N}(x) = \alpha_1 F_X(\sigma_1(x)) + (1 - \alpha_1) F_X(\sigma_2(x)) \quad (\text{A-3})$$

Defining $\sigma_{2 \rightarrow 1}(x) \equiv \sigma_1^{-1}[\sigma_2(x)]$, as the signal bidder-type 1 would need to observe to choose the same bid as bidder-type 2, rewrite A-3 so as to focus on the distribution of valuations by:

$$\begin{aligned} F_{S,N}(\sigma_1^{-1}(x)) &= \alpha_1 F_X(x) + (1 - \alpha_1) F_X(\sigma_1^{-1}[\sigma_2(x)]) \\ &= \alpha_1 F_X(x) + (1 - \alpha_1) F_X(\sigma_{2 \rightarrow 1}(x)) \end{aligned}$$

This expression recovers the distribution over valuations as:

$$F_X(x) = \frac{1}{\alpha_1} F_{S,N}(\sigma_1^{-1}(x)) - \frac{1 - \alpha_1}{\alpha_1} F_X(\sigma_{2 \rightarrow 1}(x)) \quad (\text{A-4})$$

Assume (wlog) that $\alpha_1 > \frac{1}{2}$, and use the equation A-4 as the basis for iteratively defining the distribution over valuations as a function of the bidding distributions and strategies, since:

$$\begin{aligned} F_X(\sigma_{2 \rightarrow 1}(x)) &= \frac{1}{\alpha_1} F_{S,N}(\sigma_1^{-1}(\sigma_{2 \rightarrow 1}(x))) - \frac{1 - \alpha_1}{\alpha_1} F_X(\sigma_{2 \rightarrow 1}(\sigma_{2 \rightarrow 1}(x))) \\ &\equiv \frac{1}{\alpha_1} F_{S,N}(\sigma_1^{-1}(\sigma_{2 \rightarrow 1}(x))) - \frac{1 - \alpha_1}{\alpha_1} F_X(\sigma_{2 \rightarrow 1}^{(2)}(x)) \end{aligned}$$

Then write the distribution over valuations as the infinite sum:

$$\begin{aligned}
F_X(x) &= \frac{1}{\alpha_1} F_{S,N}(\sigma_1^{-1}(x)) - \frac{1-\alpha_1}{\alpha_1^2} F_{S,N}(\sigma_1^{-1}(\sigma_{2 \rightarrow 1}(x))) + \frac{(1-\alpha_1)^2}{\alpha_1^2} F_X(\sigma_{2 \rightarrow 1}^{(2)}(x)) \\
&= \frac{1}{\alpha_1} F_{S,N}(\sigma_1^{-1}(x)) + \sum_{i=1}^{\infty} (-1)^i \frac{(1-\alpha_1)^i}{\alpha_1^{i+1}} F_{S,N}(\sigma_1^{-1}(\sigma_{2 \rightarrow 1}^{(i)})) \tag{A-5}
\end{aligned}$$

This last sum converges since $\sum_{t=1}^T \frac{(1-\alpha_1)^t}{\alpha_1^{t+1}} \rightarrow_{T \rightarrow \infty} C < \infty$ and $0 \leq F_{S,N}(\sigma_1^{-1}(\sigma_{2 \rightarrow 1}^{(i)})) \leq 1$.

Extending the argument to a general number of bidder-types is straightforward (though it requires somewhat cumbersome notation) when there is a dominant bidder-type, with the only challenge being to prove that the sum in equation A-5 converges. When there is no dominant bidder-type, one can be constructed as a mixed-strategy of $K - 1$ bidder-types' level- k strategies with the argument proceeding inductively.

Not for Publication: Web Appendix

W.1 Strategic Beliefs and Equilibrium Identification

Here I present a slightly more direct identification argument under the equilibrium behavioral model than is contained in Guerre, Perrigne, and Vuong (2000). Dropping the t subscript unless needed for clarity, individual i 's payoff is:

$$U_i(X_i, s_1, \dots, s_N) = (X_i - s_i) 1_{\{s_i > \max_{j \neq i} s_j\}}$$

Conditional on X_i , the independence of valuations (and, consequently, of bids) implies player i 's expected utility from the bid s_i is:

$$E[U_i(X_i, s_1, \dots, s_N) | X_i] = (X_i - s_i) Pr\{s_i > \max_{j \neq i} s_j\}$$

Equilibrium analysis typically begins by hypothesizing a behavior for other players, solving for player i 's best response to this behavior, and then finding a fixed point where everyone's behavior is consistent with rational beliefs. This analysis ensures that, first, all players are best responding to beliefs and, second, that those beliefs reflect the true joint distribution of behavior and valuations. While the first feature links the individuals' bids to bidder valuations, it is the second feature of equilibrium that links the econometrician's information set to the player's information set, providing the key to establishing identification. Given the empirical distribution over bids, symmetry and independence imply the true probability that player i will win the auction with a bid of s_i is given by the cumulative distribution of the highest bid among $N - 1$ independent competing bidders: $Pr\{s_i > \max_{j \neq i} s_j\} = F_S(s_i)^{N-1}$. Since equilibrium requires player i 's beliefs to match this empirical distribution over bids, the expected utility that player i seeks to maximize in equilibrium is:

$$E[U_i(X_i, s_1, \dots, s_N) | X_i] = (X_i - s_i) F_S(s)^{N-1}$$

The first order conditions for optimal behavior establish identification of the IPV auction model by recovering the true valuation for any bid directly from the bid's value and the

distribution over bids, F_S . These first order conditions give the inverse bidding function:

$$x_i = s_i + \frac{F_S(s_i)}{(N-1)f_S(s_i)} \quad (\text{W.1})$$

By focusing on beliefs, the proof highlights a key feature of equilibrium behavior: player i 's expected utility depends only on their own independent private valuation and other player's actions, which are i.i.d. with a distribution contained in the econometrician's information set. Many existing results regarding the identification of auctions, notably those that incorporate parametric risk aversion (such as Campo, Guerre, Perrigne, and Vuong (Forthcoming)) as well as the asymmetric bidder model studied by Brendstrup and Paarsch (2006), can be similarly proved directly by exploiting this property. This more direct proof technique could be extended to state a set of sufficient conditions that can be applied to establish identification in a range of games with incomplete information, including principal-agent problems, coordination and search games.

W.2 Identification Details for Level- k Behavioral Types

This appendix walks through each of the types in Crawford and Iriberri (2007a)'s level- k behavioral model, characterizing identification as a corollary to Theorem 1 in the main text.

W.2.1 Trivial Identification & Non-Identification Results

For several of the behavioral types, in particular the level-0 player-types, identification is either trivial or impossible. Whenever there is a single truthful bidder, that bidder's distribution over bids will stochastically dominate all other bidders' distributions over bids. Consequently, if the econometrician observes enough information to identify each individual bidder's distribution over bids, the distribution over valuations is trivially identified by the distribution over bids. On the other hand, if the population consists entirely of purely random bidders, whose behavior is independent of their latent valuations, then identification is obviously impossible regardless of the information obtained about the distribution of bids.

Lemma W.1 (Level-0 Trivial Identification & Non-Identification)

A *Suppose there is at least one $L0_T$ bidder-type in the population and the econometrician observes the distribution over bids for each individual, then the distribution over valuations is identified.*

B *Suppose all bidders in the population are $L0_R$ bidder-types, then even if the econometrician observes the distribution of bids for each individual, the distribution over valuations is unidentified.*

In the example from section 1.4, the Random Level 1 ($L1_R$) bidder-type's strategy is linear in his valuation, presenting a setting where identification is only slightly more complicated than the truthful level zero bidder-type. Since his beliefs, and consequently his behavior, are invariant to changes in the distribution over valuations, the $L1_R$ bidder-type provides a simple context for formalizing the level- k behavioral model free of identification problems. Following Crawford and Iriberry (2007a) in adopting Krishna (2002)'s notation, denote the maximum bid submitted by players other than player i by the random variable Y_i . The $L1_R$ bidding function solves:

$$\sigma_{L1_R}(x) = \operatorname{argmax}_{\sigma: X \rightarrow S} (X_i - \sigma(X_i)) F_{Y_i}(\sigma(X_i))$$

This maximization problem yields first order conditions given in Crawford and Iriberry (2007a)'s, Equation 14:

$$(X_i - \sigma(X_i)) f_{Y_i}(\sigma(X_i)) - F_{Y_i}(\sigma(X_i)) = 0 \tag{W.2}$$

The $L1_R$ bidder believes Y_i to be the maximum of $(N - 1)$ uniformly distributed random variables, behaving as if Y_i has cdf and pdf $F_{Y_i}(s) = \frac{s^{N-1}}{\bar{x}^{N-1}}$ and $f_{Y_i}(s) = \frac{(N-1)s^{N-2}}{\bar{x}^{N-1}}$, respectively. These beliefs yield the best responding bidding function:

$$\sigma_{L1_R}(X_i) = \frac{N-1}{N} X_i$$

Since the $L1_R$ bidding function is a linear transformation of the valuation, identification is immediate. With distribution over bids denoted $F_{S,L1_R}$, the cdf and pdf for valuations are:

$$\begin{aligned} F_X(x) &= F_{S,L1_R}(\sigma_{L1_R}(x)) = F_{S,L1_R}\left(\frac{N-1}{N}X_i\right) \\ f_X(x) &= f_{S,L1_R}(\sigma_{L1_R}(x)) \frac{N-1}{N} = f_{S,L1_R}\left(\frac{N-1}{N}X_i\right) \frac{N-1}{N} \end{aligned} \quad (\text{W.3})$$

Hence, a single-step estimation procedure can infer the distribution over valuations by shifting and scaling the estimated distribution over bids.¹³

Corollary W.1 (Identification of $L1_R$ Bidder-Type Valuations) *Suppose the econometrician observes the distribution over bids, $F_{S,L1_R}$, from a homogeneous population of $L1_R$ bidder-types, then the distribution over valuations, F_X , is identified.*

W.2.2 Identification of Higher-Order Bidder-Types

The higher-order bidder-types best respond to beliefs that their opponents take into account the distribution over valuations in choosing their bid. These bidder-types follow more sophisticated strategies than described in the previous section, with the distinction becoming particularly salient when reserve prices are considered in section 5.

W.2.2.1 Truthful Level 1 ($L1_T$) and Random Level 2 ($L2_R$)

The Truthful Level 1 ($L1_T$) bidder-type best responds to the belief that other players submit bids exactly equal to their valuations. As such, in equation W.2, the $L1_T$ bidder behaves as if Y_i is the maximum of $(N-1)$ random variables drawn from the distribution for valuations, with cdf and pdf $F_{Y_i}(s) = F_X(s)^{N-1}$, and, $f_{Y_i}(s) = \frac{1}{N-1}F_X(s)^{N-2}f_X(s)$, respectively. Crawford and Iriberry (2007a) show these beliefs yield first order conditions:

$$X_i = \sigma_{L1_T}(X_i) + \frac{F_X(\sigma_{L1_T}(X_i))}{(N-1)f_X(\sigma_{L1_T}(X_i))} \quad (\text{W.4})$$

¹³While this result does not follow directly from Theorem 1, it is stated as a corollary since the $L1_R$ bidder type can be modeled as if he believes his opponents' bidding functions are given by: $\sigma_{L0_R}(x) = \bar{x} * F_X^{-1}(x)$.

Here, the conditions on F_X in Theorem 1 ensure that the implicitly-defined bidding function, $\sigma_{L1_T}(X_i)$, is well-defined, uniformly continuous, and strictly increasing in X_i .

Identification in this setting presents the first challenging result in the paper. The inverse bidding function in equation W.4 characterizes the identified set as containing any distribution over valuations consistent with the observed distribution over bids. However, the inverse bidding function itself depends on the true F_X . For any distribution over valuations, F^* , subject to the regularity conditions in Theorem 1, define:

$$\sigma_{*,L1_T}^{-1}(s) = s + \frac{F^*(s)}{(N-1)f^*(s)}$$

The observational equivalence of F_X and F_* under the $L1_T$ behavioral model then requires:

$$F_X(\sigma_{L1_T}^{-1}(s)) = F_s(s) = F^*(\sigma_{*,L1_T}^{-1}(s)) \quad (\text{W.5})$$

The identification argument establishes that any distribution F^* satisfying this relationship must be identical to F_X almost everywhere through a pair of contradictions. These contradictions provide the template for the general proof of theorem 1, exploiting the regularity conditions and the implied properties of the bid-shading behavior for any bidder.

First, define $\epsilon_1 \equiv \inf \{x : F_X(x) \neq F^*(x)\} > 0$ and $\epsilon_2 \equiv \inf \{x > \epsilon_1 : F_X(x) = F^*(x)\} > \epsilon_1$ so that for $y \in [0, \epsilon_1)$, $F_X(y) = F^*(y)$ and, as such, the inverse bidding functions are identical to one another in this region, i.e., $\sigma^{-1}(y) = \sigma_*^{-1}(y)$. Note that $\sigma^{-1}(y)$ is strictly greater than y away from the origin, continuous, and strictly increasing, so there is some $\tilde{y} < \epsilon_1$ with $\sigma^{-1}(\tilde{y}) \in (\epsilon_1, \epsilon_2)$. Then, $F_X(\sigma^{-1}(\tilde{y})) \neq F^*(\sigma_*^{-1}(\tilde{y}))$, contradicting W.5. As such, since the bid-shade is non-negative, continuous, and zero at the origin, any candidate distribution over valuations satisfying the condition W.5 must either be identical to the true distribution or differ from the true distribution starting at the origin.

Now, suppose the distributions F_X and F^* diverge immediately from the origin and, wlog, that $F_X(x) > F^*(x)$ for $x \in (0, \epsilon)$ where $\epsilon \equiv \sup \{x : F_X(x) > F^*(x)\}$. In this case, the condition in equation W.5 demands that $\sigma^{-1}(x) < \sigma_*^{-1}(x)$. However, there must come a point in $(0, \epsilon)$ where the distribution F^* begins ‘‘catching up’’ with F_X , i.e., where $f_X(x) < f^*(x)$. But since $F_X(x) > F^*(x)$, these inequalities imply $\frac{F_X(x)}{f_X(x)} > \frac{F^*(x)}{f^*(x)}$, contradicting

the requirement that $\sigma^{-1}(x) < \sigma_*^{-1}(x)$. Here, monotonicity of bidding couples with the definition of F_X and F^* as the integral of f_X and f^* , respectively, to establish that any two distributions that diverge immediately from the origin cannot be observationally equivalent.

The identification argument for the Random Level 2 bidder-type closely mirrors the analysis of the Truthful Level 1 bidder-type since both bidder-types best respond to the belief that other players' bids are a linear transformation of their valuation. Incorporating the known constant Jacobian term, Crawford and Iriberry (2007a) show the first order condition [W.4](#) above becomes:

$$X_i = \sigma_{L2R}(X_i) + \frac{F_X\left(\frac{N}{N-1}\sigma_{L2R}(X_i)\right)}{Nf_X\left(\frac{N}{N-1}\sigma_{L2R}(X_i)\right)}$$

The identification proof generalizes immediately, as stated in the following corollary:

Corollary W.2 (Identification of $L1_T$ and $L2_R$ Bidder-Types)

- A** *Suppose the econometrician observes the distribution over bids, $F_{S,L1_T}$, from a homogeneous population of $L1_T$ bidder-types, then the distribution over valuations, F_X , is identified.*
- B** *Suppose the econometrician observes the distribution over bids, $F_{S,L2_R}$, from a homogeneous population of $L2_R$ bidder-types, then the distribution over valuations, F_X , is identified.*

W.2.2.2 Truthful Level 2 ($L2_T$)

Identification in the $L2_T$ case is complicated by the lack of a closed-form solution for the $L1_T$ bidding strategy. Crawford and Iriberry (2007a) characterize the first order condition [W.2](#) for the $L2_T$ bidder-type as:

$$(X - \sigma_{L2T}(X)) f_X(\sigma_{L1T}^{-1}(\sigma_{L2T}(X))) \frac{d\sigma_{L1T}^{-1}(s)}{ds} \Big|_{s=\sigma_{L2T}(X)} - F_X(\sigma_{L1T}^{-1}(\sigma_{L2T}(X))) = 0 \quad (\text{W.6})$$

While no closed-form solution exists for the $L1_T$ bidding function, equation [W.4](#) gives the inverse of the $L1_T$ bidding function, with corresponding derivative:

$$\frac{d\sigma_{L1T}^{-1}(s)}{ds} = 1 + \frac{f_x(s)^2 - F_X(s) f'_X(s)}{(N-1) f_X(s)^2} = \frac{N}{N-1} - \frac{F_X(s) f'_X(s)}{(N-1) f_X(s)^2}$$

Substituting this identity into equation W.6 and rearranging gives the $L2_T$ inverse bidding function:

$$X_i = \sigma_{L2_T}(X_i) + \frac{F_X \left(\sigma_{L2_T}(X_i) + \frac{F_X(\sigma_{L2_T}(X_i))}{(N-1)f_x(\sigma_{L2_T}(X_i))} \right)}{f_X \left(\sigma_{L2_T}(X_i) + \frac{F_X(\sigma_{L2_T}(X_i))}{(N-1)f_x(\sigma_{L2_T}(X_i))} \right) \left(N - \frac{F_X(\sigma_{L2_T}(X_i))f'_X(\sigma_{L2_T}(X_i))}{f_X(\sigma_{L2_T}(X_i))^2} \right)}$$
(W.7)

Here, the distribution over bids for the $L2_T$ bidder-type also depends on the derivative of the pdf for true valuations, introducing a new potential source for confounding identification. However, given condition 4 in Theorem 1, the bracketed expression in the denominator of equation W.7 is positive and bounded away from zero, with implicit differentiation establishing the bidding equation $\sigma_{L2_T}(X_i)$ as monotonic in valuation X_i .

The identified set is characterized by the consistency requirement in equation W.5 from the previous section applied to the first order conditions in W.7. Here the argument for identification has little structural difference, except the requisite continuity conditions apply to the higher order derivatives of the $L1_T$ bidder-type's strategy. Nonetheless, the approach of establishing contradictions through analyzing the bid shade is effectively unchanged.

Corollary W.3 (Identification of $L2_T$ Bidder-Type) *Suppose the econometrician observes distribution over bids, $F_{S,L2_T}$, from a homogeneous population of $L2_T$ bidder-types, then the distribution over valuations, F_X , is identified.*

W.2.2.3 General Level- k

The general level- $(k - 1)$ bidding strategy is a continuous function of the bidder's signal and $(k - 1)$ derivatives of the pdf over valuations, hence each iteration of the cognitive hierarchy requires another continuous derivative of the distribution over valuations as indicated in assumption L1.3 in lemma 1. The regularity conditions in lemma 1 ensure this bidding strategy has derivatives that exist, are bounded, and continuous, giving rise to a continuous, monotonic level- k bidding function. The general level- k first order conditions from Crawford

and Iriberry (2007a) are:

$$\begin{aligned} & (X - \sigma_{Lk_\tau}(X)) (N - 1) f_X \left(\sigma_{L(k-1)_\tau}^{-1}(\sigma_{Lk_\tau}(X)) \right) \frac{d\sigma_{L(k-1)_\tau}^{-1}(s)}{ds} \Big|_{s=\sigma_{Lk_\tau}(X)} \\ & = F_X \left(\sigma_{L(k-1)_\tau}^{-1}(\sigma_{Lk_\tau}(X)) \right) \end{aligned}$$

Rearranging this equation gives the inverse bidding function that characterizes consistency required for the distribution over bids to be generated by the distribution over valuations:

$$X = \sigma_{Lk_\tau}(X) + \frac{F_X \left(\sigma_{L(k-1)_\tau}^{-1}(\sigma_{Lk_\tau}(X)) \right)}{(N - 1) f_X \left(\sigma_{L(k-1)_\tau}^{-1}(\sigma_{Lk_\tau}(X)) \right) \frac{d\sigma_{L(k-1)_\tau}^{-1}(s)}{ds} \Big|_{s=\sigma_{Lk_\tau}(X)}} \quad (\text{W.8})$$

The observational equivalence arguments developed in previous sections are applied to the relationship in equation W.8 in the proof in appendix 1.

W.3 Semi-Nonparametric Maximum Likelihood Consistent Estimation Details

This section presents additional background to the proof establishing consistent estimation using the Legendre polynomial sieve.

W.3.1 Parametric Likelihood Functions

Suppose bidder i is of the k th bidder-type, so the observed bid, $s_i = \sigma_k(x_i)$ and, equivalently, $x_i = \sigma_k^{-1}(s_i)$. Then the cumulative likelihood of having observed a bid less than s_i , conditional on the true parameter vector θ is:

$$F_{S,k}(s_i; \theta) = F_X(\sigma_k^{-1}(s_i; \theta); \theta)$$

Differentiating with respect to s_i and applying the Jacobian of the inverse bidding function,

the likelihood of observing a bid equal to s_i for bidder-type k is:

$$f_{S,k}(s_i; \theta) = \frac{f_X(\sigma_k^{-1}(s_i; \theta); \theta)}{\sigma'_k(\sigma_k^{-1}(s_i; \theta); \theta)}$$

To characterize this mixture structure, define an indicator that the i th bid was chosen by a k -type bidder by $d_{ik} = 1_{\{\tau(i)=k\}}$, which can be stacked into the vector $d_i = (d_{i1}, \dots, d_{iK})'$. The likelihood for the i th observation conditional on the i th bidder's type can then be stated in either of two forms:

$$f(s_i; d_i, \theta) = \sum_{k=1}^K d_{ik} f_{S,k}(s_i; \theta) = \prod_{k=1}^K f_{S,k}(s_i; \theta)^{d_{ik}}$$

Since the bidder's type is independent of their valuation, the distribution for the type generating the i th bid is a multinomial random variable with distribution:

$$p_k(\theta) = Pr(\tau(i) = k) = \prod_{k=1}^K p_k^{d_{ik}}$$

Combining these two results gives the distribution of the i th bid conditional on the distribution over valuations:

$$f(s_i; \theta) = \prod_{k=1}^K p_k^{d_{ik}} f_{S,k}(s_i; \theta)^{d_{ik}} \quad (\text{W.9})$$

The unconditional likelihood of observing all bids then provides the basis for the expected log likelihood that serves as the criterion function for maximum likelihood estimation:

$$\mathcal{L}_T(\theta; s_1, \dots, s_T) = \prod_{i=1}^T \prod_{k=1}^K p_k^{d_{ik}} f_{S,k}(s_i; \theta)^{d_{ik}} \quad (\text{W.10})$$

$$\hat{\Psi}_T(\theta; s_1, \dots, s_T) = \sum_{i=1}^T \sum_{k=1}^K d_{ik} \ln p_k + d_{ik} \ln f_{S,k}(s_i; \theta) \quad (\text{W.11})$$

The analog population criterion function can then be represented by:

$$\Psi(p, \theta) = E[d_k \ln p_k + d_k \ln f_{S,k}(S; \theta) | p, \theta] \quad (\text{W.12})$$

In a panel sample with repeated observations of an individual following a constant bidding

strategy, the likelihood has additional structure reflecting the additional information about that bidder's type. Denoting the sample of T_i bids for individual i by $S_i = \{S_{i,1}, \dots, S_{i,T_i}\}$, the probability of observing a sample of bids $s_i = \{s_{i,1}, \dots, s_{i,T_i}\}$ given $\tau(i) = k$ is:

$$f_{S,k}(s_i; \theta) = \prod_{t=1}^{T_i} \frac{f_X(\sigma_k^{-1}(s_{i,t}); \theta)}{\sigma'_k(\sigma_k^{-1}(s_{i,t}); \theta)}$$

Since this is the only basic definition that changes, the likelihood for the full bidding sample remains as stated in equation W.10.

In the benchmark setting, the econometrician only observes the winning bid, as in a Dutch descending auction where the auction ends once the winning bidder claims the object at the announced price. The distribution of winning bids is given by the distribution for the maximum bid, which will depend on the actual mixture of types in each round. As such, computing the unconditional distribution for winning bids requires summing over all possible mixtures of the K bidder-types. Defining $F_{S_N,k}$ as the distribution over bids for the k -th bidder-type in an auction with N participating bidders, the resulting distribution is most readily stated in terms of cumulative densities:

$$F_{W_N}(w; \theta) = \sum_{n_1=0}^N \sum_{n_2=0}^{N-n_1} \dots \sum_{n_{K-1}=0}^{N-\sum_{k=1}^{K-2} n_k} \binom{N}{n_1} \binom{N-n_1}{n_2} \dots \binom{N-\sum_{k=1}^{K-2} n_k}{n_{K-1}} \prod_{k=1}^K p_k^{n_k} F_{S_N,k}(w; \theta)^{n_k} \tag{W.13}$$

Note that the distribution over winning bids is a continuous polynomial in the distribution over bidder-types implying the expected value of the winning bid, or the expected revenue from the auction, is also continuous in the distribution over bidder-types.

W.3.2 Legendre Sieve Space

This appendix provides background on the Legendre Polynomial sieve and basic results due to Bierens (2008) and Bierens and Song (2010) establishing the sieve space as a compact metric space. Note that computation using these distributions is numerically challenging that is greatly aided by some clever algorithms described in Bierens (2008). These techniques are

not reproduced here but readers are referred to Bierens (2008) for implementation guidance when using these procedures.

Definition W.1 (Legendre Polynomials) Legendre polynomials $\rho_n(x)$ of order $n \geq 2$ are defined recursively by the formula:

$$\rho_n(u) = \frac{\sqrt{2n-1}\sqrt{2n+1}}{n} (2u-1) \rho_{n-1}(u) - \frac{(n-1)\sqrt{2n+1}}{n\sqrt{2n-3}} \rho_{n-2}(u)$$

with $\rho_0(u) = 1, \rho_1(u) = \sqrt{3}(2u-1)$

To adapt the Legendre polynomials to density estimation, Bierens (2008) shows that any density function $h(u)$ on $[0, 1]$ can be represented as:

$$h(u) = \frac{\left(1 + \sum_{j=1}^{\infty} \delta_j \rho_j(u)\right)^2}{1 + \sum_{j=1}^{\infty} \delta_j^2}, \text{ where, } \sum_{j=1}^{\infty} \delta_j^2 < \infty \quad (\text{W.14})$$

While the unit-interval support addresses measurability issues regarding the arguments of equation W.14, some additional constraints are needed to ensure the parameters δ_j are well-behaved to ensure compactness for the space of density functions. This constraint takes the following form:

Lemma W.2 (Constrained Legendre Polynomial Sieve Space (Bierens & Song))

Let \mathcal{D} be the space of density functions $h(u)$ of the form W.14 where, for some a priori chosen constant $c > 0$, the parameters δ_j satisfy:

$$|\delta_j| \leq c \left(1 + \sqrt{j} \ln j\right)^{-1}, j = 1, 2, 3, \dots$$

Then with the L^1 metric, \mathcal{D} is a compact metric space. Also, letting $G(v)$ and $g(v)$ be as in Assumption 4, the space

$$\mathcal{D}(G) = \{f(v) = h(G(v))g(v), h \in \mathcal{D}\}$$

of densities on $[0, \overline{M}]$ with the L^1 metric is also a compact metric space. Further, the corresponding spaces of absolutely continuous distribution functions on $[0, 1]$ and $[0, \overline{M}]$, respectively,

$$\mathcal{H} = \left\{ H(u) = \int_0^u h(z) dz, h \in \mathcal{D} \right\} \mathcal{F} = \left\{ F(v) = \int_0^v f(z) dz, f \in \mathcal{D}(\mathcal{G}) \right\}$$

with the L^1 metric are compact metric spaces.

Proof. Bierens (2008) Theorems (8) and (9) and Bierens and Song (2010), Lemmas (5), (6), and (7). ■

The formal consistency argument requires verifying three key convergence properties for the finite-sample criterion function. The uniform strong law of large numbers for the criterion function is a special case of the general uniform strong law of large numbers in Artstein and Wets (1995). Convergence of the criterion function at its optimum follows from the USLLN and continuity of the criterion function at the optimum. Lastly, Bierens and Song (2010) show that the criteria function's optimum over a sequence of constrained sieve spaces converges to the the global optimum of the unconstrained sieve space. These results are summarized in the following lemma.

Lemma W.3 (Convergence Results for Upper Semicontinuous Criterion Functions)

a. Let Θ be a compact metric space with metric $\rho(\theta_1, \theta_2)$, and let $\Psi_t(\theta), t = 1, 2, \dots, T, \dots$ be a sequence of i.i.d. random, real valued, upper semicontinuous functions on Θ . If, in addition, for each $\theta_0 \in \Theta$, there exists an open set $Q_0 \subset \Theta$ and a constant $\xi_0 < \infty$ such that

$$\sup_{\theta \in Q_0} \Psi_1(\theta) < \xi_0 \text{ a.s.}$$

then

$$\sup_{\theta \in \Theta} \left| \frac{1}{N} \sum_{j=1}^N \Psi_j(\theta) - \Psi(\theta) \right| \rightarrow 0 \text{ a.s.}$$

b. Suppose further that $\Psi(\theta)$ is an upper semicontinuous real function on Θ , define $\hat{\Psi}_N(\theta) = \frac{1}{N} \sum_{j=1}^N \Psi_j(\theta)$, and let $\hat{\theta}_N = \arg \max_{\theta \in \Theta} \hat{\Psi}_N(\theta)$ and $\theta_0 = \arg \max_{\theta \in \Theta} \Psi(\theta)$. Then for

$N \rightarrow \infty,$

$$\Psi(\hat{\theta}_N) \rightarrow \Psi(\theta_0) \text{ a.s.}$$

If θ_0 is unique, then $\rho(\hat{\theta}_N, \theta_0) \rightarrow 0$ a.s.

c. Let $\{\Theta_n\}_{n=0}^\infty$ be an increasing sequence of compact subspaces of Θ for which the computation of

$$\hat{\theta}_{n,N} = \arg \max_{\theta \in \Theta_n} \hat{\Psi}_N(\theta) \tag{W.15}$$

is feasible. Suppose that for each $\theta \in \Theta$ there exists a sequence $\theta_n \in \Theta_n$ such that $\lim_{N \rightarrow \infty} n_N = \infty$, and denote the sieve estimator involved by $\tilde{\theta}_N = \hat{\theta}_{n,N}$. Then $\rho(\tilde{\theta}_N, \theta_0) \rightarrow 0$ a.s.

Proof. The conditions in statement (a) are strictly stronger than the sufficient conditions for the uniform strong law of large numbers in Artstein and Wets (1995) Theorem 2.3 but are easily verified to apply to the level- k model. The remaining results follow immediately from combining this strong law of large numbers with the arguments in Bierens and Song (2010), Theorems (1) - (3). ■

These convergence results provide the basis for SNP-consistent estimation of the level- k auction model. Given compactness results for the Legendre polynomial sieves in lemma W.2, let the distribution over valuations be indexed by H , the equivalent distribution over the unit interval from assumption 4 that admits a Legendre polynomial representation. Redefine $\theta = [p', H]'$ to join the distribution over types and distribution over valuations into a single parameter vector belonging to a metric space, Θ , with the metric:

$$\rho(\theta_1, \theta_2) = \max \left[\max |p_1 - p_2|, \sup_{0 \leq u \leq 1} |H_1(u) - H_2(u)| \right] \tag{W.16}$$

The population criterion function is given by $\Psi(\theta)$ from equation W.12, with the sample counterpart $\hat{\Psi}_T(\theta)$ from equation W.11.

W.3.3 A Generalized EM Algorithm

Given the types generating each bid, d_{ik} , the log likelihood function W.10 is:

$$\ln \mathcal{L} \left(\theta; \{s_i\}_{i=1}^T, \{d_{ik}\}_{i=1}^T \}_{k=1}^K \right) = \sum_{i=1}^T \sum_{k=1}^K d_{ik} \ln f_{S,k}(s_i; \theta) + \sum_{i=1}^T \sum_{k=1}^K d_{ik} \ln p_k$$

Though this likelihood function is not directly observable, it can be approximated by taking the expectation over the unobservable d_{ik} parameters to get an expected log likelihood:

$$E \left[\ln \mathcal{L} \left(\theta; \{s_i\}_{i=1}^T, \{d_{ik}\}_{i=1}^T \}_{k=1}^K \right) \right] = \sum_{i=1}^T \sum_{k=1}^K E[d_{ik}] \ln f_{S,k}(s_i; \theta) + \sum_{i=1}^T \sum_{k=1}^K E[d_{ik}] \ln p_k$$

To compute this expected likelihood, initialize the process with an a priori guess for the distributional parameters θ_0 and the distribution over types p_0 . Then estimate $\hat{z}_{ik,0} = E_0[d_{ik}]$, which is the probability that bid s_i is drawn from the distribution of bids for the k th bidder-type. This probability is a straightforward application of Bayes' rule given by a formula from the mixture-of-types models of Stahl and Wilson (1994), Stahl and Wilson (1995) and Costa-Gomes, Crawford, and Broseta (2001):

$$\hat{z}_{ik,0} = p_k(s_i; \theta, p) = \frac{p_k(\theta) f_{S,k}(s_i; \theta)}{\sum_{\kappa \in \mathcal{K}} p_\kappa(\theta) f_{S,\kappa}(s_i; \theta)} \quad (\text{W.17})$$

The Expectation step in the Expectation Maximization (EM) algorithm then approximates the above log likelihood by:

$$E_0 \left[\ln \mathcal{L} \left(\theta; \{s_i\}_{i=1}^T, \{d_{ik}\}_{i=1}^T \}_{k=1}^K \right) \right] = \sum_{i=1}^T \sum_{k=1}^K \hat{z}_{ik} \ln f_{S,k}(s_i; \theta) + \sum_{i=1}^T \sum_{k=1}^K \hat{z}_{ik} \ln p_k$$

The Maximization step in the EM algorithm then chooses the parameter vector θ_1 and distribution over types p_1 to maximize this expected log likelihood and proceeds to iterate between the Expectation and Maximization steps until these distributional estimates converge. Since the parameters governing the distribution over bids for fixed bidder-type does not depend on p , the maximization problem can be separated into two pieces. First, the updated distribution over types must be the average probability that a bidder is drawn from

that type. That is:

$$p_{k,1} = \frac{1}{T} \sum_{i=1}^T \hat{z}_{ik,0} \quad (\text{W.18})$$

$$\theta_1 = \arg \max_{\theta} \sum_{i=1}^T \sum_{k=1}^K \hat{z}_{ik} \ln f_{S,k}(s_i; \theta) \quad (\text{W.19})$$

Computationally, the Expectation step and the first piece of the Maximization step in the EM algorithm is quite fast, even when numerical methods are used to compute the equilibrium bidding and inverse bidding functions and their associated derivatives. However, the Maximization step that requires generating new estimates for θ is quite cumbersome due to its role in computing equilibrium bidding functions. As such, repeated application of the EM algorithm until the algorithm converges is computationally infeasible and requires a great deal of redundant calculations. Further, the convergence for the parameter estimates occurs much more quickly than convergence for the distribution over behavioral types.

To address this issue, I introduce a Generalized EM algorithm that proceeds as follows:

Algorithm 1 *Generalized Expectation Maximization Algorithm*

Step 0: *Initiate model with a priori guesses for p_0 and θ_0 , choose tolerance δ , set $p_{0a} = p_0$.*

Step 1: *Expectation Step: Use equation W.17 to compute $\hat{z}_{ik,0}$.*

Step 2: *Partial Maximization Step:*

Step 2a: *Use equation W.18 to compute p_1 .*

Step 2b: *If $\|p_0 - p_1\| > \delta$, set $p_0 = p_1$ and return to Step 1.*

Step 3: *Complete Maximization Step:*

Step 3a: *Choose θ_1 to maximize the expected log likelihood formula in equation W.19.*

Step 3b: *Use equation W.18 to compute p_1*

Step 3c: *If $\|p_{0a} - p_1\| + \|\theta_0 - \theta_1\| > \delta$, set $p_0 = p_1$, $p_{0a} = p_1$, $\theta_0 = \theta_1$ and return to Step 1.*

Generalized EM (GEM) algorithms are well-known tools for addressing maximum likelihood estimation problems. Instead of completely maximizing the likelihood function in each of the iterations of the GEM algorithm, the algorithm chooses a set of parameters that ensures the likelihood's value increases with each iteration. As such, the GEM algorithm satisfies the key condition for convergence to the optimum presented in Casella and Berger (2001), Theorem 7.2.20. However, it is possible for the GEM algorithm to fail to converge, as I do not establish formal almost sure convergence results for the algorithm as presented for a class of general stochastic optimization procedures in Biscarat (1994), Chan and Ledolter (1995) and Sherman, Dalal, and Ho (1999).

Web Appendix 4: Level- k Bidding with Reserve Prices

This appendix develops two key results regarding level- k bidding with reserve prices that closely parallel existing results for equilibrium behavior. First, it presents the general level- k bidding strategy when there is a reserve price in the auction. Second, it characterizes the effect of uncertain competition on level- k bidding behavior. The appendix closes with a result characterizing the expected revenue in a level- k auction as a weighted average of the expected revenues conditional on the composition of the bidding population.

W4.1 Certain Competition

When bidders know the number of participating bidders in the auction, counterfactual bidding behavior treats the distribution over valuations for participating bidders conditional on the valuation being greater than the reserve price. As such, denoting the level $(k - 1)$ bidding strategy when the reserve price is r by $\sigma_{L(k-1),r}(X)$ the inverse bidding function from equation W.8 incorporates this information:

$$X = \sigma_{Lk_\tau}(X) + \frac{F_X\left(\sigma_{L(k-1),\tau}^{-1}\left(\sigma_{Lk_\tau}(X)\right)\right) - F_X(r)}{f_X\left(\sigma_{L(k-1),\tau}^{-1}\left(\sigma_{Lk_\tau}(X)\right)\right) \frac{d\sigma_{L(k-1),\tau}^{-1}(s)}{ds}\Big|_{s=\sigma_{Lk_\tau}(X)}} \quad (\text{W.20})$$

As in equilibrium, the effect of the reserve price on level- k bidding behavior is to shift the differential equation defining bid shades to initialize at the reserve price rather than the

minimum valuation (which is here set to zero for exposition). This effect is apparent in the estimated bidding strategies displayed in figure 8's depiction based on estimates from the USFS timber auction data.

W4.2 Uncertain Competition

When bidders know only the number of potentially participating bidders, \bar{N} , but not the actual number of potential bidders with valuations that exceed the reserve price, N , they face an uncertain amount of competition in the auction. In another parallel to a well-known equilibrium result stated in Krishna (2002), the bidding strategy for level- k bidders with uncertain competition is a weighted average of the bidding strategies with a fixed number of participating bidders. To establish this result, first suppose the number of bidders varies exogenously and denote $Pr\{N = n\} = q_n$. Then given the level $(k - 1)$ bidding strategy, the expected utility from the level- k bidder-type's bid is:

$$E[U(X_i, s_1, \dots, s_N) | X_i] = \sum_{n=1}^{\bar{N}} (X_i - s_i) q_n Pr\{s_i > s_{-i}\}^{n-1}$$

Substituting $Pr\{s_i > s_{-i}\} = F_X\left(\sigma_{L(k-1)\tau}^{-1}(s_i)\right)$ and taking first order conditions gives the inverse bidding function as:

$$\begin{aligned} X &= \sum_{n=1}^{\bar{N}} \frac{(n-1) q_n F_X\left(\sigma_{L(k-1)\tau}^{-1}(s_i)\right)}{\sum_{m=1}^{\bar{N}} (m-1) q_m F_X\left(\sigma_{L(k-1)\tau}^{-1}(s_i)\right)} \left\{ s_i + \frac{F_X\left(\sigma_{L(k-1)\tau}^{-1}(s_i)\right)}{(n-1) f_X\left(\sigma_{L(k-1)\tau}^{-1}(s_i)\right) \frac{d\sigma_{L(k-1)\tau}^{-1}(\xi)}{d\xi} \Big|_{\xi=s_i}} \right\} \\ &= \sum_{n=1}^{\bar{N}} \omega_n(s_i) \sigma_{k_n}^{-1}(s_i) \end{aligned} \quad (\text{W.21})$$

In the case of a binding reserve price, q_n is the probability that n bidders will have a valuation exceeding the reserve price given that player i 's valuation is above the reserve price, which is given by:

$$q_n = \binom{\bar{N}-1}{n-1} F_X(r)^{N-n} (1 - F_X(r))^{n-1}$$

So the inverse bidding function with endogenous participation is:

$$X = \sum_{n=1}^{\bar{N}} \omega_{n,r}(s_i) \sigma_{k_{n,r}}^{-1}(s_i)$$

W4.3 Expected Revenues

First, consider the seller's expected revenue when participating bidders know the number of bidders participating in each auction. In this case, the cdf for the winning bid is given by F_{W_N} in equation . Ex ante, the seller does not know the number of bidders whose valuations will exceed the reserve price but only the number of potential bidders, \bar{N} . Accounting for this uncertainty, the cdf for the seller's revenue is then:

$$F_{W_{\bar{N}}}(w; \theta) = \sum_{N=0}^{\bar{N}} \binom{\bar{N}}{N} F_X(r)^{\bar{N}-N} (1 - F_X(r))^N F_{W_N}(w; \theta) \quad (\text{W.22})$$

The key feature of this formula for analyzing expected revenues is that, since F_{W_N} is a continuous polynomial in the distribution over types, the expected revenue to the seller is continuous in the distribution over types. Further, because the behavior of each bidder and bidder-type is independent of one another, the distribution over bidder-types that maximizes and minimizes these expected revenues corresponds to the homogeneous populations of bidder-types that individually maximize and minimize the sellers' expected revenue, respectively. This result is stated in theorem 6 in the main text.